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ELEMENTARY WORKS

BY THE REV. JOHN HIND, M.A.

LATE FELLOW AND TUTOR OF
SIDNEY SUSSEX COLLEGE, CAMBRIDGE.

1.

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THE ELEMENTS

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WITH THE

NATURE AND PROPERTIES OF LOGARITHMS,

AND THE

CONSTRUCTION AND USE OF MATHEMATICAL TABLES.

DESIGNED FOR THE USE OF STUDENTS
IN THE UNIVERSITY.

BY

JOHN HIND, M.A. F.C.P.S. F.R.A.S.

LATE FELLOW AND TUTOR OF SIDNEY SUSSEX COLLEGE,
CAMBRIDGE.

FIFTH EDITION.

CAMBRIDGE:
DEIGHTON, BELL & CO.
LONDON: BELL AND DALDY, 186, FLEET STREET.
1855.

Cambridge: Printed at the University Press.

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THE following pages are designed to enable the Student to become acquainted with the *Principles* and *Applications* of *Trigonometry*, without requiring of him any thing more than a knowledge of the *Elements of Geometry*, and a facility in the common operations of *Arithmetic* and *Algebra*.

The first four chapters of the *Plane Trigonometry*, contain what is most essentially necessary for the study of the best modern works on *Science;* and pains have been taken to render the demonstrations of the fundamental propositions as concise as possible, without at all affecting their logical accuracy.

The remaining chapters take a more extensive and *analytical* view of the Nature and Applications of *Trigonometry* and *Logarithms*, the particulars of which, as well as of those in *Spherical Trigonometry*, may be seen in the *Table of Contents.*

In the *Appendices* will be found many interesting *Theorems* and *Problems* with their *Solutions*, as well as Collections of *Examples for Practice* with their *Answers* annexed; and the Reader, who may be desirous of arriving at a proficiency in Mathematical Science, will scarcely neglect to avail himself of some of their suggestions.

CAMBRIDGE,
May 21, 1855.

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TRIGONOMETRY.

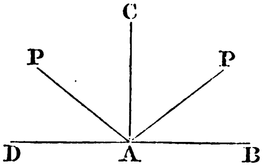
CHAPTER I.

DEFINITIONS AND PRELIMINARY OBSERVATIONS.

ARTICLE I. DEFINITION.

TRIGONOMETRY in its original acceptation is the *Mensuration* of the sides and angles of plane rectilineal Triangles: but it here includes the *General Doctrine* of *Angles;* their *Relations* to one another; and their *Dependence* upon the *Lines* by which they are formed, or with which they are in any manner connected.

2. DEF. An *Angle* is the mutual *Inclination* of two straight lines to one another, which meet together, and are not in the same direction; or, it is the *Divergence* of two straight lines from the same point: thus, in the annexed diagram, where *A* is called the *Angular Point,* each of the magnitudes *BAP* is a rectilineal angle, the term *Inclination* being used in an extended sense so as to comprise them both.



3. Def. When the line AC is in such a situation that the angles BAC and CAD are equal to each other, each of them is called a *Right Angle*.

Also, the angle BAP is said to be *acute* or *obtuse*, according as it is *less* or *greater* than a right angle.

4. Def. In the estimation of angular magnitude, the right angle BAC is regarded as the *primitive* angle, by means of which and its parts, all other angles are valued and compared. The right angle is supposed to be divided into 90 equal angles, called *Degrees;* each degree is supposed to be divided into 60 equal angles called *Minutes;* each minute into 60 equal angles called *Seconds;* and so on, according to a *sexagesimal* subdivision: but it is usual in practice to express all denominations inferior to seconds as decimals of a second.

These subdivisions are denoted by the characters $^\circ$, $'$, $''$, $'''$, &c., placed above the line a little to the right of the numbers to which they belong: thus, $42^\circ\, 37'\, 28''\, 21'''$ represents 42 *degrees,* 37 *minutes,* 28 *seconds*, 21 *thirds ;* but it may be written $42^\circ\, 37'\, 28''.35$, since 21 thirds reduced to the decimal of a second is .35.

In this system, a Degree, which is expressed by 1°, is the *Angular Unit,* or the *Unit* of *Angular Measure.*

5. Def. The excess of *one* right angle above any angle containing A degrees is called the *Complement* of the angle A, and is written $90^\circ - A^\circ$.

Thus, the complement of $34^\circ\, 15' = 90^\circ - 34^\circ\, 15' = 55^\circ\, 45'$;

and the complement of $23^\circ\, 27'\, 53''.67 = 90^\circ - 23^\circ\, 27'\, 53''.67$
$= 66^\circ\, 32'\, 6''.33$.

Hence, also, in any *right-angled* triangle, each of the *acute* angles is the complement of the other, because all the angles are together equal to two right angles, or to 180°.

In the diagram, PAC is the complement of BAP.

6. Def. The excess of *two* right angles above the angle A° is called the *Supplement* of A°, and is written $180^\circ - A^\circ$.

Thus, the supplement of $44^\circ\,16' = 180^\circ - 44^\circ\,16' = 135^\circ\,44'$; and the supplement of $173^\circ\,3'\,13''.81 = 180^\circ - 173^\circ\,3'\,13''.81 = 6^\circ\,56'\,46''.19$.

Since all the angles of a plane triangle are together equal to two right angles, it follows that each of the angles of every triangle is the supplement of the *sum* of the two others.

In the diagram, PAD is the supplement of BAP.

7. Def. Most of the modern Continental Mathematicians divide the right angle into 100 equal angles called *Grades;* each grade into 100 equal angles termed *Minutes;* each minute into 100 equal angles called *Seconds;* and so on, according to a *centesimal* subdivision: thus, $35^g\,29^{`}\,53^{``}$ is read 35 *grades*, 29 *minutes*, 53 *seconds*, and is equal to .352953 of a right angle.

This scale avoids all *vulgar fractions* in the operations upon numerical angular magnitudes: and it is scarcely necessary to remark, that the complement and supplement of A^g will be represented by $100^g - A^g$ and $200^g - A^g$ respectively.

8. Cor. Hence the numbers of *Degrees*, &c., and *Grades*, &c., expressing the magnitude of any angle are readily compared.

Let E and F denote the *numbers* of English and Foreign degrees, expressed decimally when necessary, contained in any angle BAP: then, since

$$\frac{\text{the angle } BAP}{\text{a right angle}} = \frac{E}{90} = \frac{F}{100}, \text{ we have}$$

$$E = \frac{90}{100}F = \frac{9}{10}F = \frac{10-1}{10}F = F - \frac{1}{10}F:$$

$$F = \frac{100}{90}E = \frac{10}{9}E = \frac{9+1}{9}E = E + \frac{1}{9}E:$$

that is, the number of English degrees is equal to the number of Foreign *diminished* by a *tenth* part: and the number of Foreign is equal to the number of English *increased* by a *ninth* part.

Examples.

$$\begin{array}{rr} \text{Let } F = & 25^{g}\,44^{\backprime}\,89^{\backprime\backprime} \\ = & 25.44890: \\ \therefore \tfrac{1}{10}F = & \underline{2.54489} \\ \therefore E = & 22.90401 \\ & \underline{60} \\ & 54.24060 \\ & \underline{60} \\ & \underline{14.43600} \end{array}$$

or, $E = 22^{0}\,54'\,14''.436$.

In the same manner

$1^{g} = 0^{0}\,54'$

$1^{\backprime} = 0'\,32''.4$

$1^{\backprime\backprime} = 0''.324$

&c. = &c.

$$\begin{array}{rr} \text{Let } E = & 31^{0}\,45'\,57'': \\ \text{then } 60) & \underline{57.0000} \\ 60) & \underline{45.9500} \\ E = & 31.765833 \\ \therefore \tfrac{1}{9}E = & \underline{3.529537} \\ \therefore F = & \underline{35.295370} \end{array}$$

or, $F = 35^{g}\,22^{\backprime}\,53^{\backprime\backprime}.7$

In the same manner

$1^{0} = 1^{g}\,11^{\backprime}\,11^{\backprime\backprime}$ &c.

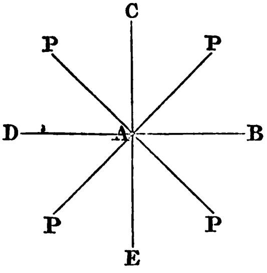
$1' = 1^{\backprime}\,85^{\backprime\backprime}.18$ &c.

$1'' = 3^{\backprime\backprime}.0864$ &c.

&c. = &c.

9. The definition of an angle above given, which is that of *Geometry*, implies that every angle must be less than *two* right angles, and it will therefore be sufficient for the consideration of the angles of triangles and other rectilineal figures in their ordinary acceptation: but the Application of Trigonometry to *Philosophical Enquiries* dependent upon other than purely geometrical Principles, has rendered it necessary to take an extended view of angular magnitude, and the following genesis of an angle will enable us to understand how *angular magnitude* may admit of indefinite increase or decrease, in the same manner as *lineal*, or *any other kind* of extension.

10. Let *BAD* be any straight line assumed at pleasure, *A* the angular point in it: then, by the continued



revolution of the line AP about the fixed point A, from its primitive position AB, in a plane passing through BAD, an angle or angular space, of any magnitude whatever, may be generated or produced.

For, if the line CAE passing through the point A cut the line BAD at right angles, and the angle generated be called A: when AP falls within the right angle BAC, called the *first* right angle, the angle A lies between 0° and 90°: when AP falls within the *second* right angle CAD, the angle A lies between 90° and 180°: when AP falls within the *third* right angle DAE, the angle A lies between 180° and 270°: and when AP falls within the *fourth* right angle EAB, the angle A lies between 270° and 360°.

Also, if the revolution of AP be further continued, the magnitude of the angle A will be still further increased; and the increase produced by every successive *complete* revolution will be *four* right angles, or 360°.

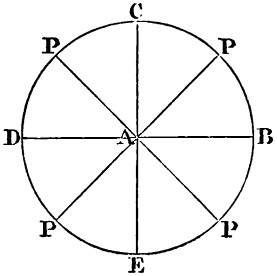
11. COR. Since, by the *progressive* revolution of the line AP from its primitive position AB, the magnitude of the angle A has been *increased:* it follows that a *regressive* revolution in the contrary direction, must have the effect of *diminishing* that angle: that is, according to the *Principles of Algebra*, if the angle BAP in the right angle BAC be considered *positive*, the angle BAP in the right angle BAE must be considered *negative:* and *positive* and *negative* angles are always understood in this sense, unless the contrary be expressed.

In Articles (5) and (6) the *Complements* and *Supplements* of angles will therefore be negative, when the angles themselves are greater than 90° and 180° respectively.

12. It is evident that 1°, the unit of angular measure adopted in Article (4), can admit of no comparison with any *lineal* unit whatever, whether it be an *inch*, a *foot*, a *yard*, &c.; inasmuch as they are altogether *heterogeneous* magnitudes, or magnitudes of *different* kinds: still however a lineal unit may be applied both to *measure* and to *determine* the magnitude of an angle, as will appear from the following articles.

13. *To find the connection between an angle, and the arc of a circle of given radius which subtends it at the centre.*

With the angular point A as a centre, and any radius AB, describe a circle: through A draw CAE at right angles to BAD: then will the circumference of the circle be divided into four equal parts BC, CD, DE, EB, called *Quadrants:* also, since by *Euclid*, Book VI.



Prop. 33, arcs have the same ratio to each other as the angles which they respectively subtend at the centre, the division of the right angle into 90 equal angles will *correspond* to the division of the quadrantal arc into 90 equal arcs. Hence arises the division of the quadrantal arc into 90, and consequently that of the whole circumference into 360 equal parts, called *Degrees* of the quadrant or circle, which are therefore the units of the *lineal* measure of the corresponding angles.

Let now $\angle BAP = A^{0}$, radius $AB = r$, and arc $BP = a$: then, since the circumference of a circle whose radius is r, is $2\pi r$, where π is the numerical magnitude 3.1415926 &c. as hereafter proved, we shall have

$$\begin{aligned} A^{0} : 90^{0} &:: BP : BC \\ &:: a : \tfrac{1}{4}2\pi r \\ &:: 2a : \pi r; \end{aligned}$$

$$\text{whence } A^{0} = \frac{90^{0} \times 2a}{\pi r} = \frac{180^{0}}{\pi} \frac{a}{r} = \frac{180^{0}}{3.1415926 \text{ \&c.}} \times \frac{a}{r}$$

$$= 57^{0}.2957795 \text{ \&c.} \times \frac{a}{r}:$$

that is, if the *magnitudes* of r and a be given in *lineal* measure, the *number* of *Degrees* in the corresponding angle A will be obtained by multiplying $57^\circ.2957795$ &c., or $57^\circ\,17'\,44''.6$ &c. by the fraction $\frac{a}{r}$, or $\frac{\text{arc}}{\text{radius}}$.

14. Cor. Since $57^\circ.2957795$ &c. is an invariable magnitude, it follows that the angle A *varies* as

$$\frac{\text{the arc } BP}{\text{the radius } AB}:$$

and the *comparison* of angles will be simplified by the adoption of the *equality*

$$A^\circ = \frac{a}{r}, \text{ or angle} = \frac{\text{arc}}{\text{radius}}:$$

where it must be remembered that the *Unit* of angular measure is now $57^\circ.2957795$ &c., instead of 1° before made use of.

Hence, the magnitude of an angle is both *measured* and *determined* by the fraction $\frac{\text{arc}}{\text{radius}}$.

15. Def. The fraction $\frac{arc}{radius}$ is called the *Circular Measure* of the corresponding angle; and if the *radius* be equal to the *lineal* unit or be represented by 1, the *arc* will be the circular measure of the angle, on the scale in which 2π, or 6.2831852 &c. is the circular measure of four right angles.

If the circular measure be $\frac{2}{3}$, we shall have

$$\begin{aligned}\text{the angle} &= \tfrac{2}{3}(57^\circ.2957795 \text{ \&c.})\\ &= 38^\circ.1971863 \text{ \&c.}\end{aligned}$$

The angular unit in circular measure $= 57^\circ.2957795$ &c. $= 206264.8$ seconds nearly: and consequently the magnitude of an angle expressed in *seconds* will be $= \frac{\text{arc}}{\text{radius}} \times 206264.8$, nearly.

If $A = 1''$, we shall therefore have the circular measure of $1'' = \frac{1}{206264.8} = .00000484813$ &c.

16. Cor. 1. The *Circular* measure is always used in *Analysis*, and it has been seen above how it may be converted into the *Arithmetical* measure, and vice versâ.

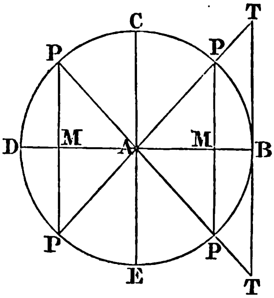
Also, since $A^0 = 57^0.2957795$ &c. $\times \frac{a}{r}$; if we suppose $a = r$, we shall have $A^0 = 57^0.2957795$ &c.; and the angular unit in circular measure is therefore the angle subtended by the arc of the circle which is equal to the radius: that is, if the radius of the circle be regarded as the *lineal* unit, we have only to multiply the angular unit in this measure $57^0.2957795$ &c. by the *number* of lineal units in the arc, to obtain the number of degrees, &c. in the corresponding angle, and *vice versâ.*

17. Cor. 2. If we denote $57^0.2957795$ &c. by r^0, we shall have $A^0 = r^0 \frac{a}{r}$, or $rA^0 = ar^0$, so that if $A = 1$, the radius r will be equal to 57.2957795 &c. times the arc: also, if $\theta = \frac{a}{r}$, we have $A^0 = \theta r^0$.

18. Cor. 3. Whenever a symbol θ expresses the circular measure of an angle, from Article (15), its *Complement* will be denoted by $\frac{1}{2}\pi - \theta$, and its *Supplement* by $\pi - \theta$: it may be remarked however that this notation is not always strictly attended to, as π and 180^0 are frequently used for each other: but the neglect of it can produce no inconvenience, because all quantities, connected by the signs + and −, are necessarily supposed to be of the same kind.

19. We have now seen how an angle may be *measured* by means of the radius of a circle whose centre is the angular point, and the arc which subtends it: and we shall next shew how that *arc,* and therefore the corresponding *angle,* may be *determined* by the use of certain *straight lines,* whose relations to each other are discovered by common Geometry, and which are termed, *Trigonometrical Functions of a Circular Arc.*

20. Def. From the centre A with any radius AB, let a circle be described, and BAD, CAE be diameters at right angles to each other: then, if BP be an arc beginning at B, and terminating at P situated in any of the four quadrants, the following definitions, of what are called the *Circular Functions,* have been adopted.



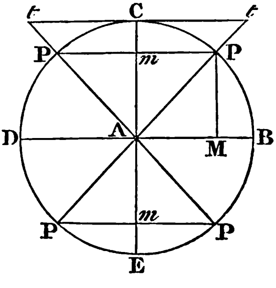
(1) The *Sine* of an arc *BP* is the straight line *PM* drawn from the end of the arc, perpendicular to the diameter passing through the beginning of it.

(2) The *Tangent* of an arc *BP* is the straight line *BT* touching the arc at the beginning, and terminated by the radius, through the end of it, produced.

(3) The *Secant* of an arc *BP* is the straight line *AT* drawn through the centre and the end of the arc, and terminated by the tangent.

(4) The *Versed Sine* of an arc *BP* is the part *BM* of the diameter, intercepted between the beginning of the arc and the sine.

Again, the arc *PC* being the *Complement* of the arc *BP*, the corresponding lines belonging to it, and distinguished by the affix *co* to the preceding, are defined as follows.



(5) The *Cosine* of an arc *BP* is the straight line *Pm* drawn from the end of the arc, perpendicular to the diameter through the end of the first quadrant.

(6) The *Cotangent* of an arc BP is the straight line Ct touching the circle at the end of the first quadrant, and terminated by the radius, through the end of the arc, produced.

(7) The *Cosecant* of an arc BP is the straight line At drawn through the centre and the end of the arc, and terminated by the cotangent.

(8) The *Coversed Sine* of an arc BP is the part Cm of the diameter, intercepted between the end of the first quadrant and the cosine.

If the arc be increased by *any number* of *entire* circumferences, all these functions will evidently remain the same.

21. Cor. Since Pm in the latter diagram is equal to AM in the former, the cosine of an arc BP is equal to the straight line AM intercepted between the centre and the sine, and may be defined accordingly.

Also, if the points B and P be joined, and a straight line be drawn perpendicular to BP from the centre, it is evident that the *Chord* BP of the arc BP is equal to *twice* the sine of *half* that arc.

22. If a denote the arc BP, these functions may be written in abbreviated forms: as, $\sin a$, $\tan a$, $\sec a$, $\text{vers}\, a$; $\cos a$, $\cot a$, $\text{cosec}\, a$, $\text{covers}\, a$; and in computation they will of course be treated as any other magnitudes expressed by means of *Algebraical* or *Arithmetical* Symbols.

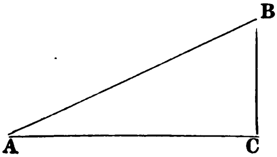
Hence, the *square* or *second* power of $\sin a$ is $(\sin a)^2$, the *cube* or *third* power of $\cos a$ is $(\cos a)^3$: but it will generally be found more convenient to *write* $\sin^2 a$, $\cos^3 a$, as the representations of these powers, although this notation has a different meaning in some cases, where it is however always particularized.

23. If the radius of the circle be given, each of these functions will in general *determine* the arcs, and therefore the corresponding angles: but inasmuch as they are not *proportional* to them, they cannot be adopted as their *measures*.

24. Def. The magnitude of an angle is independent of the lengths of the lines by which it is formed;

and, accordingly, since the ratios of the sides of a triangle remain the same whilst its angles are not altered, the magnitude of an angle may be determined by means of the following *Ratios* called *Angular Functions*, or *Trigonometrical Functions of an Angle*, borrowing their Character and Nomenclature from the lines in article (20), the circle being dispensed with altogether.

Let the angle BAC be denoted by A, where B is *any* point in the line AB, and BC is drawn perpendicular to AC; then, we have the following *Definitions.*



(1) $$\text{Sin } A = \frac{BC}{AB}, \text{ or } BC = AB \sin A:$$

(2) $$\text{Cos } A = \frac{AC}{AB}, \text{ or } AC = AB \cos A:$$

(3) $$\text{Tan } A = \frac{BC}{AC}, \text{ or } BC = AC \tan A:$$

(4) $$\text{Cot } A = \frac{AC}{BC}, \text{ or } AC = BC \cot A:$$

(5) $$\text{Sec } A = \frac{AB}{AC}, \text{ or } AB = AC \sec A:$$

(6) $$\text{Cosec } A = \frac{AB}{BC}, \text{ or } AB = BC \operatorname{cosec} A.$$

The versed sine, coversed sine, and chord have no existence in this System, because it is independent of the radius of a circle; but from Analogy, $1 - \cos A$ is generally expressed by vers A, $1 - \sin A$ by covers A, and $2 \sin \frac{1}{2} A$ by chord A.

It is evident that these functions are all *Arithmetical* magnitudes: and that they will be of the same values if the angle be increased by any multiple of four right angles.

Also, an angle may generally be *determined* by means of its sine, cosine, &c.; because they are the ratios of the sides of a right-angled triangle, which ratios always remain the same, whilst the angles are not altered.

25. Cor. Since $A + B = 90^\circ$, because C is a right angle, we have $B = 90^\circ - A$: and therefore by the last article:

(1) $$\text{Sin}\, B = \frac{AC}{AB} = \cos A, \text{ or } \sin(90^\circ - A) = \cos A:$$

(2) $$\text{Cos}\, B = \frac{BC}{AB} = \sin A, \text{ or } \cos(90^\circ - A) = \sin A:$$

(3) $$\text{Tan}\, B = \frac{AC}{BC} = \cot A, \text{ or } \tan(90^\circ - A) = \cot A:$$

(4) $$\text{Cot}\, B = \frac{BC}{AC} = \tan A, \text{ or } \cot(90^\circ - A) = \tan A:$$

(5) $$\text{Sec}\, B = \frac{AB}{BC} = \text{cosec}\, A, \text{ or } \sec(90^\circ - A) = \text{cosec}\, A:$$

(6) $$\text{Cosec}\, B = \frac{AB}{AC} = \sec A, \text{ or } \text{cosec}\,(90^\circ - A) = \sec A.$$

Hence, to pass from any function of an angle to that of its complement, we have only to change the places of sine, cosine: tangent, cotangent: &c.: respectively.

26. *To investigate the fundamental relations subsisting among the trigonometrical functions of an angle.*

(1) $$\text{Sin}^2 A + \cos^2 A = \left(\frac{BC}{AB}\right)^2 + \left(\frac{AC}{AB}\right)^2 = \frac{BC^2 + AC^2}{AB^2}$$
$$= \frac{AB^2}{AB^2} = 1;$$

whence, $\sin^2 A = 1 - \cos^2 A$, and $\cos^2 A = 1 - \sin^2 A$.

(2) $$\text{Sec}^2 A = \left(\frac{AB}{AC}\right)^2 = \frac{AB^2}{AC^2} = \frac{AC^2 + BC^2}{AC^2} = 1 + \frac{BC^2}{AC^2}$$
$$= 1 + \left(\frac{BC}{AC}\right)^2 = 1 + \tan^2 A;$$ whence, $\tan^2 A = \sec^2 A - 1$.

(3) $$\text{Cosec}^2 A = \left(\frac{AB}{BC}\right)^2 = \frac{AB^2}{BC^2} = \frac{BC^2 + AC^2}{BC^2} = 1 + \frac{AC^2}{BC^2}$$
$$= 1 + \left(\frac{AC}{BC}\right)^2 = 1 + \cot^2 A;$$ whence, $\cot^2 A = \text{cosec}^2 A - 1$.

(4) $$\text{Tan } A = \frac{BC}{AC} = \frac{BC}{AB} \div \frac{AC}{AB} = \sin A \div \cos A = \frac{\sin A}{\cos A}.$$

(5) $$\text{Cot } A = \frac{AC}{BC} = \frac{1}{\tan A} = \frac{\cos A}{\sin A};$$

whence, $\tan A \cot A = 1$.

(6) $$\text{Sec } A = \frac{AB}{AC} = \frac{AB}{AB} \div \frac{AC}{AB} = 1 \div \cos A = \frac{1}{\cos A}.$$

(7) $$\text{Cosec } A = \frac{AB}{BC} = \frac{AB}{AB} \div \frac{BC}{AB} = 1 \div \sin A = \frac{1}{\sin A}.$$

By reason of the results which have just been established, the cotangent, secant, and cosecant are sometimes *defined* to be the reciprocals of the tangent, cosine, and sine respectively; and by substitution, each of these functions may be expressed in a variety of different forms, the processes constituting what is called the *Arithmetic of Sines.*

Examples.

(1) $$\text{Cos}^2 A = \frac{1}{\sec^2 A} = \frac{1}{1 + \tan^2 A}:$$

$$\text{and } \sin^2 A = \cos^2 A \tan^2 A = \frac{\tan^2 A}{1 + \tan^2 A}.$$

(2) $$\text{Cos}^2 A = 1 - \sin^2 A = 1 - \frac{1}{\text{cosec}^2 A} = \frac{\text{cosec}^2 A - 1}{\text{cosec}^2 A}:$$

$$\text{and } \sin^2 A = 1 - \cos^2 A = 1 - \frac{1}{\sec^2 A} = \frac{\sec^2 A - 1}{\sec^2 A}.$$

(3) $$\text{Sin}^2 A = 1 - \cos^2 A = 1 - (1 - \text{vers } A)^2$$

$$= 1 - 1 + 2 \text{ vers } A - \text{vers}^2 A = 2 \text{ vers } A - \text{vers}^2 A:$$

$$\text{and } \cos^2 A = 1 - \sin^2 A = 1 - (1 - \text{covers } A)^2$$

$$= 1 - 1 + 2 \text{ covers } A - \text{covers}^2 A = 2 \text{ covers } A - \text{covers}^2 A.$$

(4) $$\text{Tan}^2 A = \frac{\sin^2 A}{\cos^2 A} = \frac{1 - \cos^2 A}{\cos^2 A}$$

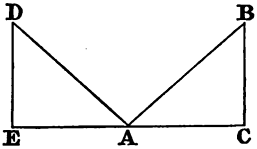
$$= \frac{1 - (1 - \text{vers } A)^2}{(1 - \text{vers } A)^2} = \frac{2 \text{ vers } A - \text{vers}^2 A}{(1 - \text{vers } A)^2}:$$

$$\text{and } \sec^2 A = 1 + \tan^2 A = 1 + \frac{2 \operatorname{vers} A - \operatorname{vers}^2 A}{(1 - \operatorname{vers} A)^2}$$

$$= \frac{(1 - \operatorname{vers} A)^2 + 2 \operatorname{vers} A - \operatorname{vers}^2 A}{(1 - \operatorname{vers} A)^2} = \frac{1}{(1 - \operatorname{vers} A)^2}.$$

27. The definitions of the *Trigonometrical Ratios*, given in Article (24), are most easily understood when the angle *CAB* is *acute;* but they are all supposed to hold good, whatever be the *magnitude* of the angle to which they belong.

Thus, the ratios for determining the angle *CAD* will be

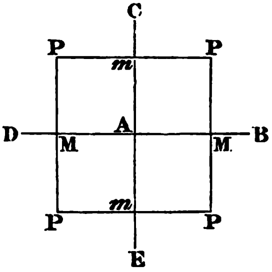


$$\text{Sin } CAD = \frac{DE}{AD}: \quad \tan CAD = \frac{DE}{AE}: \quad \sec CAD = \frac{AD}{AE}:$$

$$\text{Cos } CAD = \frac{AE}{AD}: \quad \cot CAD = \frac{AE}{DE}: \quad \operatorname{cosec} CAD = \frac{AD}{DE}.$$

Similarly, for all other positions in which the *generating line,* or the *hypothenuse* of the right-angled triangle, may happen to be situated: and it is easily shewn that the relations exhibited in Article (26) are true in all cases, which the student may *verify* for himself.

28. It will appear upon examination of the trigonometrical ratios in connection with the diagrams for the



various situations of the generating line, that their values may be *equal*, though the angles themselves are *different:* and to avoid ambiguity in such cases, some of the lines concerned are supposed to be *affected* with the *Algebraical Signs* + and –, according to a principle which we shall now proceed to explain.

Let DAB be a straight line, along which, from any fixed point A in it, different distances are to be measured and estimated: then, if M be any point in AB, we have

$$AM = AB - MB,$$

which, since AB is greater than MB, is evidently a positive quantity: but if M be situated in AD, then MB is greater than AB, and the corresponding value of AM becomes negative, and we have

$$AM = MB - AB = -(AB - MB):$$

whence, if lines measured from the fixed point A toward the *right* be called *positive*, lines measured from the same point A towards the *left* must be termed *negative*. In the same manner, if lines measured from A *upwards* along AC be positive, those measured from A *downwards* along AE will be negative.

The converse is manifestly true: and of course the same *affections* belong to all *parallel* lines drawn in the same *direction:* but if the points A, P be joined, the line AP will remain *unaffected*, as being drawn in none of the specified directions, and must consequently be treated as a *mere* arithmetical magnitude in all computations.

Ex. If A be less than 90° and $CAD = 90^\circ + A$ in the diagram of Article (27), it will easily be seen, that the *algebraical signs* of $\sin(90^\circ + A)$ and $\operatorname{cosec}(90^\circ + A)$ are +, whilst those of the other functions are –. Similarly, the signs of $\tan(180^\circ + A)$ and $\cot(180^\circ + A)$ are +, whilst the signs of the rest are –: and it is evident that the signs of the *sine* and *cosine* will always be sufficient for deciding those of the other ratios, by means of (4), (5), (6), (7), of Article (26).

29. The positive sign + being applied to all the ratios for an angle in the *first* quadrant, we shall have directly the following Synopsis of algebraical signs for the *four* quadrants.

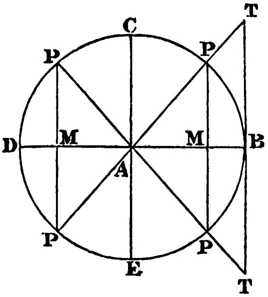
	1	2	3	4
Sine................	+	+	−	−
Cosine	+	−	−	+
Tangent	+	−	+	−
Cotangent	+	−	+	−
Secant	+	−	−	+
Cosecant	+	+	−	−
Versed sine	+	+	+	+
Coversed sine	+	+	+	+

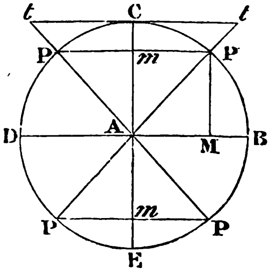
In the system of *Circular Functions*, the *secant* and *cosecant* are not drawn in the same directions as any of the rest, but their *algebraical signs* are easily derived from the equalities,

$$\sec a = \frac{r^2}{\cos a} \text{ and } \operatorname{cosec} a = \frac{r^2}{\sin a}:$$

and are therefore identical with those of the *cosine* and *sine* respectively. Also, *chord* $a = 2 \sin \frac{1}{2} a$ is positive, since $\frac{1}{2} a$ is always terminated in the *first* or *second* quadrant.

30. The values of the *Trigonometrical Ratios* at the extremities of the four right angles, as well as the magnitudes of the corresponding *Circular Functions* at those of the four quadrants when the radius of the circle is represented by 1, will be found by means of the diagrams to be as follow.





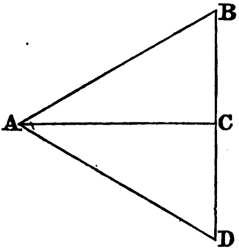
	B	*C*	*D*	*E*
Sine................	0	1	0	1
Cosine	1	0	1	0
Tangent	0	∞	0	∞
Cotangent	∞	0	∞	0
Secant	1	∞	1	∞
Cosecant...........	∞	1	∞	1
Versed sine	0	1	2	1
Coversed sine	1	0	1	2

This article and the preceding one taken together give both the *magnitudes* and *algebraical signs* of the corresponding functions: and though it be desirable that they should *all* be committed to memory, those of most frequent occurrence are here repeated:

$$\text{Sin } 0^\circ = 0: \quad \sin 90^\circ = 1: \quad \sin 180^\circ = 0:$$
$$\text{Cos } 0^\circ = 1: \quad \cos 90^\circ = 0: \quad \cos 180^\circ = -1:$$
$$\text{Tan } 0^\circ = 0: \quad \tan 90^\circ = \infty: \quad \tan 180^\circ = 0.$$

31. *To prove that sin* (– A) = – *sin* A, *and cos* (– A) = *cos* A.

Let the angle CAB be denoted by A; then, agreeably



to what was said in Article (11), the angle CAD, which is equal in *magnitude* to CAB, will be denoted by $-A$: and by the definitions, we have

$$\sin(-A) = \frac{CD}{AD} = -\frac{CB}{AB} = -\sin A:$$

$$\cos(-A) = \frac{AC}{AD} = \frac{AC}{AB} = \cos A.$$

Therefore, by means of Article (26), we have

$$\tan(-A) = \frac{\sin(-A)}{\cos(-A)} = \frac{-\sin A}{\cos A} = -\frac{\sin A}{\cos A} = -\tan A.$$

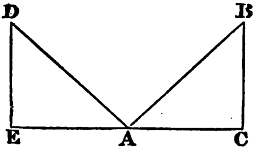
and the values of the other functions of $-A$ are immediately determined by the *Rule of Signs:* thus,

$$\cot(-A) = -\cot A: \sec(-A) = \sec A: \operatorname{cosec}(-A) = -\operatorname{cosec} A:$$
$$\operatorname{vers}(-A) = 1 - \cos A: \text{ covers}(-A) = 1 + \sin A.$$

Hence, the ratios for A and $-A$ are equal in magnitude, but different in algebraical sign, except the cosine and secant which are equal altogether, and will lead to an ambiguity in the determination of the angle.

32. *To prove that sin* $(180^0 - \text{A}) =$ *sin* A, *and cos* $(180^0 - \text{A}) = -$ *cos* A.

In the following diagram, suppose $CAB = A$;



and $CAD = 180^0 - A$: then will $EAD = 180^0 - CAD$

$$= 180^0 - 180^0 + A = A = CAB:$$

and the angles at C and E being right angles, the triangles CAB, EAD are similar:

$$\therefore \sin(180^0 - A) = \frac{DE}{AD} = \frac{BC}{AB} = \sin A,$$

the lines ED and CB being parallel to each other, and measured in the *same* direction from the line CAE:

$$\text{also, } \cos(180^0 - A) = \frac{AE}{AD} = -\frac{AC}{AB} = -\cos A,$$

the lines AE and AC being measured in *opposite* directions from the angular point A.

The values of all the other functions of $180^0 - A$, in terms of A, may be deduced from these two, by means of Article (26): thus,

$$\tan(180^0 - A) = -\tan A: \cot(180^0 - A) = -\cot A:$$
$$\sec(180^0 - A) = -\sec A: \operatorname{cosec}(180^0 - A) = \operatorname{cosec} A:$$
$$\operatorname{vers}(180^0 - A) = 1 - \cos(180^0 - A) = 1 + \cos A,$$

which is sometimes called the *suversed* sine of A, and is written suvers A: and covers $(180^0 - A) = 1 - \sin A$.

Hence, the ratios for A and $180^0 - A$ are equal in magnitude but have different algebraical signs, except the sine and cosecant which are equal in all respects, and this circumstance may cause an ambiguity as before.

33. *To find the connection between the Angular and Circular Functions of the same denominations.*

If A denote the angle, and a the corresponding arc, we see by the diagram in page (9), that

$$\frac{\sin a}{r} = \frac{MP}{AP} = \sin A:$$

and $\therefore \sin a = r \sin A$: or, $\sin a = \sin A$, if r be assumed to be the lineal unit: and similarly of the others.

Whence it follows, that the circular functions which are *lines*, may be derived from the angular functions which are *numbers*, by multiplying the radius by the latter, and *vice versâ:* and consequently an angular function is connected with the corresponding circular function, in the same way as the angular measure is with the circular measure.

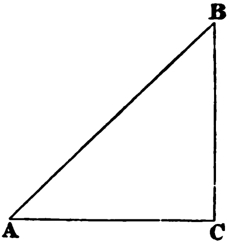
34. COR. The angular functions may therefore be *adapted to the radius r*, by substituting $\frac{\sin a}{r}$, $\frac{\cos a}{r}$, &c. in the places of $\sin A$, $\cos A$, &c. respectively: and this is effected in *practice* by retaining the same letter, and rendering all the terms of the expressions, in which the functions are involved, *homogeneous*, by the introduction of r and its powers as factors: thus, since $\sin^2 A + \cos^2 A = 1$, we have to the radius r,

$$\left(\frac{\sin A}{r}\right)^2 + \left(\frac{\cos A}{r}\right)^2 = 1; \text{ or, } \sin^2 A + \cos^2 A = r^2: \text{ \&c.}$$

35. We will conclude this chapter with a few instances of the application of the Principles which have been already explained, the results of some of which will very frequently occur to the Student.

(1) To find the sine, cosine, tangent, and cotangent of 45^0.

Let BAC be a triangle right-angled at C, having



$$\angle BAC = 45^0, \text{ and } \therefore \angle ABC = 90^0 - 45^0 = 45^0 = \angle BAC:$$

from which we have $AC = BC$:

$$\text{whence, } \sin 45^0 = \frac{BC}{AB} = \frac{AC}{AB} = \cos 45^0:$$

but $AB^2 = BC^2 + AC^2 = 2BC^2 = 2AC^2$, by *Euclid* I. 47;

$$\therefore AB = \sqrt{2}BC = \sqrt{2}AC, \text{ and } \frac{BC}{AB} = \frac{AC}{AB} = \frac{1}{\sqrt{2}}:$$

$$\text{or, } \sin 45^0 = \frac{1}{\sqrt{2}} = \cos 45^0:$$

$$\text{and } \therefore \tan 45^0 = \frac{\sin 45^0}{\cos 45^0} = 1 = \frac{\cos 45^0}{\sin 45^0} = \cot 45^0.$$

Hence, if A be less than 45^0, B is greater than 45^0: and $\therefore AC$ is greater than BC, and $\frac{AC}{AB}$ is greater than $\frac{BC}{AB}$: or, $\cos A$ is greater than $\sin A$: and *vice versâ.*

(2) To find the sines, cosines, tangents, and cotangents of 30^0 and 60^0.

Let BAD be an equilateral triangle, for which, see the diagram of Article (31): draw AC perpendicular to BD, and this will bisect both the angle BAD and the side BD: then $\angle BAD = \frac{1}{3}$ of two right angles $= 60^0$, and $\therefore \angle BAC = 30^0$: whence we have

$$\sin 30^0 = \frac{BC}{AB} = \frac{\frac{1}{2}BD}{AB} = \frac{\frac{1}{2}AB}{AB} = \frac{1}{2} = \cos 60^0:$$

$$\therefore\ \cos 30^0 = \sqrt{1 - \sin^2 30^0} = \sqrt{1 - \tfrac{1}{4}} = \sqrt{\tfrac{3}{4}} = \frac{\sqrt{3}}{2} = \sin 60^0:$$

$$\text{also, } \tan 30^0 = \frac{\sin 30^0}{\cos 30^0} = \frac{1}{\sqrt{3}} = \cot 60^0:$$

$$\text{and } \tan 60^0 = \frac{\sin 60^0}{\cos 60^3} = \sqrt{3} = \cot 30^0.$$

Hence, $2 \sin 30^0 = 1$, and ∴ in the *Circular System* of Trigonometry, the chord of 60° is equal to the radius.

(3) Given $2 \sin A = \tan A$, to find the values of A.

Here, $2 \sin A = \dfrac{\sin A}{\cos A}$, and ∴ $\cos A = \tfrac{1}{2} = \cos 60^0$: or $A = 60^0$, answers the condition: but since both members have been divided by $\sin A$, $\sin A = 0$, and ∴ $A = 0^0$, and $A = 180^0$, are also solutions.

(4) Given $\tan A + \cot A = 4$, to find $\tan A$.

Here, multiplying both sides by $\tan A$, we have

$$\tan^2 A + 1 = 4 \tan A:$$

$$\therefore\ \tan^2 A - 4 \tan A = -1:$$

completing the square by the ordinary method, we find

$$\tan^2 A - 4 \tan A + 4 = 3:$$

$$\therefore\ \tan A - 2 = \pm\sqrt{3}, \text{ and } \tan A = 2 \pm \sqrt{3}:$$

$$\text{that is, } \tan A = 2 + \sqrt{3}, \text{ and } \tan A = 2 - \sqrt{3}.$$

(5) *Given the sine and cosine of* A, *to find the sine and cosine of* 180° + A.

In the diagram of Article (20), if $A = BAP$ be *less* than a right angle, and PA be produced *from* P, we shall have by the definitions in Article (24),

$$\sin(180^0 + A) = \frac{MP}{AP} \text{ in the \textit{third} right angle,}$$

$$= -\frac{MP}{AP} \text{ in the \textit{first} right angle,}$$

$$= -\sin A:$$

$$\cos(180^\circ + A) = \frac{AM}{AP} \text{ in the } \textit{third} \text{ right angle,}$$

$$= -\frac{AM}{AP} \text{ in the } \textit{first} \text{ right angle,}$$

$$= -\cos A.$$

Similarly, if A be greater than 90° but less than 180°, the same results will be obtained.

Again, by Article (24), we have seen that

$\sin(360^\circ + A) = \sin A$, and $\cos(360^\circ + A) = \cos A$:

and taking this in connection with the preceding, we conclude, that if an angle be increased by 180°, the sine and cosine retain their *magnitudes* but change their *signs:* but if it be increased by 360°, both the *magnitudes* and *signs* are unaltered.

Hence, the *magnitudes* of all the other ratios for $180^\circ + A$ and $360^\circ + A$ will be the same as those of A: and the *algebraical signs* of the former may be determined by the relations established in Article (26).

(6) Since, $\sin(180^\circ - A) = \sin A$, and $\cos(180^\circ - A) = -\cos A$; if we change A into $-A$, and therefore $-A$ into A, we shall have

$$\sin(180^\circ + A) = \sin(-A) = -\sin A,$$

$$\cos(180^\circ + A) = -\cos(-A) = -\cos A,$$

which are both correct by the last subdivision: and thus we conclude, that the results established in Article (32), are true, whether A be a *positive* or *negative* angle.

Since, $\sin(180^\circ + A) = -\sin A$, and $\cos(180^\circ + A) = -\cos A$; if we change A into $180^\circ + A$, we shall have

$$\sin(360^\circ + A) = -\sin(180^\circ + A) = \sin A,$$

$$\cos(360^\circ + A) = -\cos(180^\circ + A) = \cos A,$$

two correct results as appears from subdivision (5): and the inference is, that the formulæ,

$\sin(180^\circ + A) = -\sin A$, and $\cos(180^\circ + A) = -\cos A$,

are true whether A be *greater* or *less* than 180°.

Hence, the observations made at the end of (5) may be depended upon, whatever be the *sign* or *magnitude* of the angle represented by the symbol A; and because

$180^\circ + A$ and $360^\circ + A$, may, by assigning proper magnitudes to A, furnish the values of all angles whatever, it will be of some advantage that they be carefully remembered.

(7) The following results are immediately deducible, if we use the circular measures π and θ for 180° and A° respectively, but they properly belong to *Analytical Trigonometry.*

If n represent any positive or negative whole number whatever, we have

$$\sin\theta = \sin(2n\pi + \theta), \text{ by Article (24):}$$

$$\text{and } \sin\theta = -\sin(-\theta) = -\sin(2n\pi - \theta), \text{ by Article (31):}$$

$$\text{that is, } \sin\theta = \pm\sin(2n\pi \pm \theta).$$

Again, $\sin\theta = \sin(\pi - \theta)$, by Article (32),

$$= \sin(2n\pi + \pi - \theta) = \sin\{(2n+1)\pi - \theta\}, \text{ by Article (24):}$$

$$\text{and } \sin\theta = -\sin\{-(\pi - \theta)\}, \text{ by Article (31),}$$

$$= -\sin\{(2n+2)\pi - (\pi - \theta)\} = -\sin\{(2n+1)\pi + \theta\},$$

by Article (24): that is,

$$\sin\theta = \pm\sin\{(2n+1)\pi \mp \theta\}.$$

Because all the angles whose sines are equal to the sine of θ are comprised in the two *forms,* $\pm(2n\pi \pm \theta)$, and $\pm\{(2n+1)\pi \mp \theta\}$, since $\sin(\pm A) = \pm\sin A$, they give the solution of the *Equation,* "$\sin\theta = m$;" and of the *Problem,* "To find general formulæ for all angles whose sines are equal to the sine of θ."

Also, $\cos\theta = \cos(2n\pi \pm \theta) = -\cos\{(2n+1)\pi \pm \theta\}$,

$$\text{and } \tan\theta = \pm\tan(2n\pi \pm \theta) = \pm\tan\{(2n+1)\pi \pm \theta\}, \text{ \&c.;}$$

which may be applied to a similar purpose.

(8) Given $\sin A = s$, to find the values of all the other trigonometrical ratios.

Since, $\sin^2 A + \cos^2 A = 1$, we have $\cos^2 A = 1 - s^2$:

$$\text{whence, } \cos A = \pm\sqrt{1 - s^2}:$$

$$\tan A = \frac{\sin A}{\cos A} = \pm\frac{s}{\sqrt{1 - s^2}}: \quad \cot A = \frac{1}{\tan A} = \pm\frac{\sqrt{1 - s^2}}{s}:$$

$$\text{also, } \sec A = \frac{1}{\cos A} = \pm\frac{1}{\sqrt{1 - s^2}}: \quad \operatorname{cosec} A = \frac{1}{\sin A} = \frac{1}{s}.$$

Of the six *fundamental* ratios, each of the rest has here been expressed in terms of the one which is given, and it appears that the *cosine, tangent, cotangent* and *secant* admit of two equal values differing in their algebraical signs: but this is easily accounted for, by considering that $s = \sin A = \sin(180^0 - A)$, by Article (32); and though the *sines* and *cosecants* of these two angles are equal *in all respects,* the other functions are not, as may be seen in the same article.

This circumstance arises entirely from the *comprehensive* character of the *Algebraical Operations:* and indeed there exist an infinite number of angles whose sines are equal to s and cosecants to $\frac{1}{s}$, but the algebraical signs of the other functions will require to be decided by an examination into the consequences of the *conventional* adoption of the signs of *Affection* which are used.

Similar results will be found from $\cos A = c$, $\tan A = t$, &c.: and in all cases where the magnitude of A itself is given, the considerations just mentioned will be indispensable.

CHAPTER II.

FUNCTIONS OF THE SUMS AND DIFFERENCES OF ANGLES, AND OF SOME OF THEIR MULTIPLES AND SUBMULTIPLES. THE TRIGONOMETRICAL CANON.

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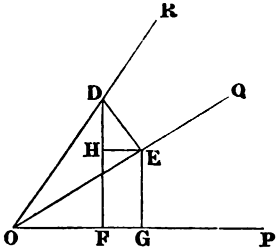
36. *To express the sine and cosine of the Sum of two angles, in terms of the sines and cosines of the angles themselves.*

Let A and B denote the angles POQ, QOR terminated in the same point O, so that $\angle\, POR = A + B$: in OR take any point D, draw



DE perpendicular to OQ; DF, EG perpendicular to OP; and EH parallel to OP, meeting FD in H:

$$\text{then, } A = \angle\, EOG = \angle\, HEO = 90^\circ - \angle\, DEH:$$

$$\therefore \sin A = \cos DEH = \frac{EH}{ED}, \text{ and } \cos A = \sin DEH = \frac{HD}{ED},$$

by the definitions of Article (24); whence,

$$\sin A = \frac{GE}{OE} = \frac{EH}{ED}, \text{ and } \cos A = \frac{OG}{OE} = \frac{HD}{ED}:$$

$$\text{also, } \sin B = \frac{ED}{OD}, \text{ and } \cos B = \frac{OE}{OD}:$$

$$\therefore \sin(A+B) = \frac{FD}{OD} = \frac{FH + HD}{OD} = \frac{GE}{OD} + \frac{HD}{OD}$$

$$= \frac{GE}{OE}\cdot\frac{OE}{OD} + \frac{HD}{ED}\cdot\frac{ED}{OD}$$

$$= \sin A \cos B + \cos A \sin B:$$

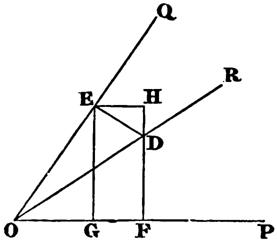
$$\text{and } \cos(A+B)=\frac{OF}{OD}=\frac{OG-FG}{OD}=\frac{OG}{OD}-\frac{EH}{OD}$$

$$=\frac{OG}{OE}\cdot\frac{OE}{OD}-\frac{EH}{ED}\cdot\frac{ED}{OD}$$

$$=\cos A\cos B-\sin A\sin B.$$

37. *To express the sine and cosine of the Difference of two angles, in terms of the sines and cosines of the angles themselves.*

Let A and B be the angles POQ, QOR, so that $POR = A-B$: take any point D in OR: draw DE perpendicular to OQ; DF, EG perpendicular to OP; and EH parallel to OP meeting FD produced in H:



then, $A = \angle EOG = \angle HEQ = 90^\circ - \angle DEH$:

$$\therefore \sin A=\cos DEH=\frac{EH}{ED}, \text{ and } \cos A=\sin DEH=\frac{HD}{ED},$$

by the definitions of (24); whence, as before,

$$\sin A=\frac{GE}{OE}=\frac{EH}{ED}, \text{ and } \cos A=\frac{OG}{OE}=\frac{HD}{ED}:$$

$$\text{also, } \sin B=\frac{ED}{OD}, \text{ and } \cos B=\frac{OE}{OD}:$$

$$\therefore \sin(A-B)=\frac{FD}{OD}=\frac{FH-HD}{OD}=\frac{GE}{OD}-\frac{HD}{OD}$$

$$=\frac{GE}{OE}\cdot\frac{OE}{OD}-\frac{HD}{ED}\cdot\frac{ED}{OD}$$

$$=\sin A\cos B-\cos A\sin B:$$

$$\text{and } \cos(A-B)=\frac{OF}{OD}=\frac{OG+FG}{OD}=\frac{OG}{OD}+\frac{EH}{OD}$$

$$= \frac{OG}{OE} \cdot \frac{OE}{OD} + \frac{EH}{ED} \cdot \frac{ED}{OD}$$

$$= \cos A \cos B + \sin A \sin B.$$

38. Cor. The transformations of the fractional forms which have been used in the last two articles in order to obtain the sines and cosines of A and B as previously exhibited, express merely the consequences arising from the composition of ratios in *Geometry*: and by comparing the formulæ investigated in them, it will appear that the latter are immediately deducible from the former, and *vice versâ*, by substituting $-B$ in the place of B, since $\sin(-B) = -\sin B$, and $\cos(-B) = \cos B$ are the results of Article (31).

Also, similar investigations may be made use of, whatever be the magnitudes of the angles A and B, due regard being paid to the algebraical signs; but the Principle of the *Permanence of Equivalent Forms* leads directly to the same results; for, if there exist any *general* formulæ for the sines and cosines of the sum and difference of two angles, they must necessarily be those which we have just found, inasmuch as they would otherwise not comprehend the cases which have here been established by particular demonstrations, and therefore could not be general.

Examples.

(1) $$\text{Sin } 15^\circ = \sin(45^\circ - 30^\circ)$$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ, \text{ by } (37),$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ :$$

$$\cos 15^\circ = \cos(60^\circ - 45^\circ)$$

$$= \cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75^\circ.$$

$$\text{Hence, } \tan 15^\circ = \frac{\sin 15^\circ}{\cos 15^\circ} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3} = \cot 75^\circ,$$

$$\text{and } \tan 75^\circ = \frac{\sin 75^\circ}{\cos 75^\circ} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3} = \cot 15^\circ.$$

(2) $\sin 135^\circ = \sin(90^\circ + 45^\circ) = \sin 90^\circ \cos 45^\circ$

$$+ \cos 90^\circ \sin 45^\circ = 1 \,.\, \frac{1}{\sqrt{2}} + 0 \,.\, \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \sin 45^\circ :$$

$$\cos 135^\circ = \cos(90^\circ + 45^\circ) = \cos 90^\circ \cos 45^\circ - \sin 90^\circ \sin 45^\circ$$

$$= 0 \,.\, \frac{1}{\sqrt{2}} - 1 \,.\, \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}} = -\cos 45^\circ.$$

(3) $\sin(90^\circ + A) = \sin 90^\circ \cos A + \cos 90^\circ \sin A$

$$= 1 \,.\, \cos A + 0 \,.\, \sin A = \cos A :$$

$$\cos(90^\circ + A) = \cos 90^\circ \cos A - \sin 90^\circ \sin A$$

$$= 0 \,.\, \cos A - 1 \,.\, \sin A = -\sin A.$$

(4) $\sin(180^\circ + A) = \sin 180^\circ \cos A + \cos 180^\circ \sin A$

$$= 0 \,.\, \cos A - 1 \,.\, \sin A = -\sin A :$$

$$\cos(180^\circ + A) = \cos 180^\circ \cos A - \sin 180^\circ \sin A$$

$$= -1 \,.\, \cos A - 0 \,.\, \sin A = -\cos A.$$

(5) $\sin\left(\frac{3\pi}{2} + \alpha\right) = \sin\frac{3\pi}{2} \cos\alpha + \cos\frac{3\pi}{2} \sin\alpha :$

but $\sin\frac{3\pi}{2} = \sin\left(\pi + \frac{\pi}{2}\right) = \sin\pi \cos\frac{\pi}{2} + \cos\pi \sin\frac{\pi}{2}$

$$= 0 \,.\, 0 - 1 \,.\, 1 = -1 :$$

$$\cos\frac{3\pi}{2} = \cos\left(\pi + \frac{\pi}{2}\right) = \cos\pi \cos\frac{\pi}{2} - \sin\pi \sin\frac{\pi}{2}$$

$$= -1 \,.\, 0 - 0 \,.\, 1 = 0 :$$

$\therefore \sin\left(\frac{3\pi}{2} + \alpha\right) = -1 \,.\, \cos\alpha + 0 \,.\, \sin\alpha = -\cos\alpha :$

similarly, $\cos\left(\frac{3\pi}{2} - \alpha\right) = 0 \,.\, \cos\alpha - 1 \,.\, \sin\alpha = -\sin\alpha :$

and the same kind of proceeding may be adopted in all quantities of this description.

39. If any one of the four fundamental formulæ established in (36) and (37) be given, the others may be deduced from it, by means of their known relations: thus, if we assume $\sin(A + B) = \sin A \cos B + \cos A \sin B$, we shall have

$$\sin(A-B)=\sin\{(180^\circ-A)+B\}=\sin(180^\circ-A)\cos B$$
$$+\cos(180^\circ-A)\sin B=\sin A\cos B-\cos A\sin B:$$
$$\cos(A+B)=\sin\{(90^\circ-A)-B\}=\sin(90^\circ-A)\cos B$$
$$-\cos(90^\circ-A)\sin B=\cos A\cos B-\sin A\sin B:$$
$$\cos(A-B)=-\cos\{(180^\circ-A)+B\}=-\cos(180^\circ-A)\cos B$$
$$+\sin(180^\circ-A)\sin B=\cos A\cos B+\sin A\sin B.$$

40. Since, $\sin(A+B)=\sin A\cos B+\cos A\sin B:$

$$\sin(A-B)=\sin A\cos B-\cos A\sin B:$$
$$\cos(A+B)=\cos A\cos B-\sin A\sin B:$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B:$$

we have, by addition and subtraction,

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B:$$
$$\sin(A+B)-\sin(A-B)=2\cos A\sin B:$$
$$\cos(A-B)+\cos(A+B)=2\cos A\cos B:$$
$$\cos(A-B)-\cos(A+B)=2\sin A\sin B:$$

and these formulæ will evidently enable us to express, in the forms of *Sums* or *Differences,* each of the *Products* in their latter members, and they deserve attention.

Ex. If $A=30^\circ$, we shall have

$$\sin(30^\circ+B)+\sin(30^\circ-B)=2\sin 30^\circ\cos B=\cos B:$$
$$\sin(30^\circ+B)-\sin(30^\circ-B)=2\cos 30^\circ\sin B=\sqrt{3}\sin B: \text{\&c.}$$

41. Cor. From the last article, we have

$$\sin(A+B)=2\sin A\cos B-\sin(A-B),$$
$$\cos(A+B)=\cos(A-B)-2\sin A\sin B, \text{\&c}:$$

which will serve to derive the functions of an angle from those of inferior angles connected with it: thus, if $A=30^\circ$, and $B=1'$;

$$\sin 30^\circ 1'=\cos 1'-\sin 29^\circ 59':$$
$$\cos 30^\circ 1'=\cos 29^\circ 59'-\sin 1': \text{\&c.}$$

that is, if the values of the sines and cosines of all angles up to 30^0 be known, those of all angles up to 60^0, will then be determined by subtraction only. By putting 45°, 60°, &c. for A or B, other useful results are obtained.

42. The operation of multiplication gives

$$\begin{aligned}
&\sin(A+B)\sin(A-B) \\
&= (\sin A\cos B+\cos A\sin B)(\sin A\cos B-\cos A\sin B) \\
&= \sin^2 A\cos^2 B-\cos^2 A\sin^2 B \\
&= \sin^2 A(1-\sin^2 B)-(1-\sin^2 A)\sin^2 B \\
&= \sin^2 A-\sin^2 A\sin^2 B-\sin^2 B+\sin^2 A\sin^2 B \\
&= \sin^2 A-\sin^2 B=(\sin A+\sin B)(\sin A-\sin B):
\end{aligned}$$

$$\begin{aligned}
\text{or, } &= (1-\cos^2 A)-(1-\cos^2 B) \\
&= \cos^2 B-\cos^2 A=(\cos B+\cos A)(\cos B-\cos A):
\end{aligned}$$

similarly, $\cos(A+B)\cos(A-B)=1-\sin^2 A-\sin^2 B$

$$\text{or, } =\cos^2 A-\sin^2 B, \text{ or, } =\cos^2 B-\sin^2 A.$$

These formulæ may be used to deduce the sines and cosines of angles from those of inferior angles, by means of the operations of common Arithmetic: thus, if $A=2^\circ$, and $B=1^\circ$, we have

$$\sin 3^\circ=\frac{(\sin 2^\circ+\sin 1^\circ)(\sin 2^\circ-\sin 1^\circ)}{\sin 1^\circ}: \text{ \&c.}$$

43. By division of the numerator and denominator, by $\cos A\cos B$, we have

$$\begin{aligned}
\frac{\sin(A+B)}{\sin(A-B)} &= \frac{\sin A\cos B+\cos A\sin B}{\sin A\cos B-\cos A\sin B} \\
&= \frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{\dfrac{\sin A}{\cos A}-\dfrac{\sin B}{\cos B}} \\
&= \frac{\tan A+\tan B}{\tan A-\tan B}:
\end{aligned}$$

similarly, $\dfrac{\cos(A+B)}{\cos(A-B)}=\dfrac{\cot A-\tan B}{\cot A+\tan B}=\dfrac{\cot A\cot B-1}{\cot A\cot B+1}.$

The object of these formulæ is to express the ratios of the *Sums* and *Differences* of functions of the latter description, by means of *single* functions.

44. *To express the sine and cosine of* 2 A *in terms of those of* A.

By means of the general formulæ of Article (36):

$$\begin{aligned}\sin 2A &= \sin (A + A)\\ &= \sin A \cos A + \cos A \sin A\\ &= 2 \sin A \cos A:\\ \cos 2A &= \cos (A + A)\\ &= \cos A \cos A - \sin A \sin A\\ &= \cos^2 A - \sin^2 A:\end{aligned}$$

$$\text{or, } = \cos^2 A - 1 + \cos^2 A = 2 \cos^2 A - 1:$$

$$\text{or, } = 1 - \sin^2 A - \sin^2 A = 1 - 2 \sin^2 A.$$

45. Cor. 1. From the last two results we have immediately,

$$2 \sin^2 A = 1 - \cos 2A, \text{ and } 2 \cos^2 A = 1 + \cos 2A:$$

and by the substitution of A and $\frac{1}{2}A$ in the places of $2A$ and A respectively, we obtain

$$\sin A = 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A, \quad 2 \sin^2 \tfrac{1}{2}A = 1 - \cos A,$$

$$2 \cos^2 \tfrac{1}{2}A = 1 + \cos A,$$

which are formulæ of great utility in the subsequent parts of the work.

Hence, we have $\dfrac{1 - \cos 2A}{1 + \cos 2A} = \dfrac{\sin^2 A}{\cos^2 A} = \tan^2 A.$

46. Cor. 2. We may easily express $\sin 2A$ and $\cos 2A$ in terms of $\tan A$: for,

$$\left.\begin{aligned}\sin 2A &= 2 \sin A \cos A\\ \cos 2A &= \cos^2 A - \sin^2 A\end{aligned}\right\}, \text{ and } 1 = \cos^2 A + \sin^2 A:$$

whence by division, we have

$$\sin 2A = \frac{2 \sin A \cos A}{\cos^2 A + \sin^2 A} = \frac{2\dfrac{\sin A}{\cos A}}{1 + \left(\dfrac{\sin A}{\cos A}\right)^2} = \frac{2 \tan A}{1 + \tan^2 A}:$$

$$\cos 2A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{1 - \left(\frac{\sin A}{\cos A}\right)^2}{1 + \left(\frac{\sin A}{\cos A}\right)^2} = \frac{1 - \tan^2 A}{1 + \tan^2 A}:$$

and these formulæ being *rational*, are frequently very convenient in practice.

47. Cor. 3. Hence, $\sin 3A = \sin(2A + A)$

$$= \sin 2A \cos A + \cos 2A \sin A$$
$$= 2 \sin A \cos^2 A + (1 - 2 \sin^2 A) \sin A$$
$$= 2 \sin A (1 - \sin^2 A) + (1 - 2 \sin^2 A) \sin A$$
$$= 3 \sin A - 4 \sin^3 A:$$

and $\cos 3A = \cos(2A + A)$

$$= \cos 2A \cos A - \sin 2A \sin A$$
$$= (2 \cos^2 A - 1) \cos A - 2 \sin^2 A \cos A$$
$$= (2 \cos^2 A - 1) \cos A - 2 (1 - \cos^2 A) \cos A$$
$$= 4 \cos^3 A - 3 \cos A,$$

two results which ought to be committed to memory.

48. *To express the sine and cosine of* A *in terms of the sine of* 2 A.

Since, $1 = \cos^2 A + \sin^2 A$, and $\sin 2A = 2 \sin A \cos A$; we have, by addition and subtraction,

$$1 + \sin 2A = \cos^2 A + 2 \sin A \cos A + \sin^2 A$$
$$= (\cos A + \sin A)^2:$$
$$1 - \sin 2A = \cos^2 A - 2 \sin A \cos A + \sin^2 A$$
$$= (\cos A - \sin A)^2:$$

whence, if $\cos A$ and $\sin A$ be positive quantities, and the square roots be extracted, we find

$$\cos A + \sin A = \sqrt{1 + \sin 2A}:$$
$$\cos A - \sin A = \pm \sqrt{1 - \sin 2A}:$$

in the latter of which the upper or lower sign must evidently be used according as $\cos A$ is greater or less than $\sin A$, or, according as A is less or greater than $45°$, by example (1) of Article (35):

$$\therefore \cos A = \tfrac{1}{2}\{\sqrt{1+\sin 2A} \pm \sqrt{1-\sin 2A}\}:$$
$$\sin A = \tfrac{1}{2}\{\sqrt{1+\sin 2A} \mp \sqrt{1-\sin 2A}\}.$$

For the explanation of the *four* values which may be found, the student is referred to the first Appendix.

These are frequently applied as *Formulæ* of *Verification*, by which the values of sin A and cos A may be put to the *test* of correctness with that of sin $2A$, and the contrary, as will be more fully explained hereafter: thus, if $2A = 60°$, we have $\sin 2A = \frac{1}{2}\sqrt{3}$; and therefore we ought to find $\cos A = \frac{1}{2}\sqrt{3}$, and $\sin A = \frac{1}{2}$, as in (35).

49. *To express the sine and cosine of* (m + 1) A *in terms of the sines and cosines of the two inferior multiples* m A, (m − 1) A, *and the cosine of* A.

Here, $\sin(m+1)A = \sin(mA + A)$
$$= \sin mA \cos A + \cos mA \sin A:$$
and $\sin(m-1)A = \sin(mA - A)$
$$= \sin mA \cos A - \cos mA \sin A:$$
$$\therefore \sin(m+1)A + \sin(m-1)A = 2 \sin mA \cos A:$$
and $\sin(m+1)A = 2\sin mA \cos A - \sin(m-1)A$:

similarly, $\cos(m+1)A = 2\cos mA \cos A - \cos(m-1)A$.

The principal use of these formulæ is in the construction of *Tables* of sines and cosines.

If $m = 1$, we have immediately,

$\sin 2A = 2\sin A \cos A$, and $\cos 2A = 2\cos^2 A - 1$,

as already proved in Article (44).

If $m = 2$, by substitution, we shall have

$$\sin 3A = 2\sin 2A \cos A - \sin A = 3\sin A - 4\sin^3 A:$$
$$\cos 3A = 2\cos 2A \cos A - \cos A = 4\cos^3 A - 3\cos A:$$

and a similar process may easily be extended to the sines and cosines of $4A$, $5A$, &c.

50. Additional formulæ of frequent occurrence.

(1) $$\frac{1-\cos 2A}{\sin 2A} = \frac{2\sin^2 A}{2\sin A\cos A} = \frac{\sin A}{\cos A} = \tan A:$$

$$\frac{1+\cos 2A}{\sin 2A} = \frac{2\cos^2 A}{2\sin A\cos A} = \frac{\cos A}{\sin A} = \cot A.$$

$$(2)\quad \frac{\text{Sec } 2A - 1}{\tan 2A} = \frac{1 - \cos 2A}{\sin 2A} = \tan A:$$

$$\frac{\sec 2A + 1}{\tan 2A} = \frac{1 + \cos 2A}{\sin 2A} = \cot A.$$

$$(3)\quad \text{Cot } A + \tan A = \frac{\cos A}{\sin A} + \frac{\sin A}{\cos A} = \frac{\cos^2 A + \sin^2 A}{\sin A \cos A}$$

$$= \frac{2}{2 \sin A \cos A} = \frac{2}{\sin 2A} = 2 \operatorname{cosec} 2A:$$

$$\cot A - \tan A = \frac{\cos A}{\sin A} - \frac{\sin A}{\cos A} = \frac{2(\cos^2 A - \sin^2 A)}{2 \sin A \cos A}$$

$$= \frac{2 \cos 2A}{\sin 2A} = 2 \cot 2A.$$

$$(4)\quad \text{Cosec } A + \cot A = \frac{1}{\sin A} + \frac{\cos A}{\sin A} = \frac{1 + \cos A}{\sin A}$$

$$= \cot \tfrac{1}{2}A:$$

$$\operatorname{cosec} A - \cot A = \frac{1}{\sin A} - \frac{\cos A}{\sin A} = \frac{1 - \cos A}{\sin A} = \tan \tfrac{1}{2}A.$$

$$(5)\quad \text{Sec } 2A + \tan 2A = \frac{1 + \sin 2A}{\cos 2A} = \frac{(\cos A + \sin A)^2}{\cos^2 A - \sin^2 A}$$

$$= \frac{\cos A + \sin A}{\cos A - \sin A} = \left(\frac{1 + \sin 2A}{1 - \sin 2A}\right)^{\frac{1}{2}}.$$

$$\text{Sec } 2A - \tan 2A = \frac{1 - \sin 2A}{\cos 2A} = \frac{(\cos A - \sin A)^2}{\cos^2 A - \sin^2 A}$$

$$= \frac{\cos A - \sin A}{\cos A + \sin A} = \frac{1 - \tan A}{1 + \tan A}.$$

51. *To find the Arithmetical Values of the sine and cosine of* 18^0: *and also of those of the sines and cosines of* 36^0 *and* 54^0 *which are immediately connected with it.*

Since, $\sin 36^0 = \cos(90^0 - 36^0) = \cos 54^0$, by Article (25): if we take A to represent 18^0, we shall have the equality $\sin 2A = \cos 3A$:

$$\text{and } \therefore\ 2 \sin A \cos A = \cos(2A + A)$$

$$= \cos 2A \cos A - \sin 2A \sin A:$$

$$\text{or, } 2\sin A = \cos 2A - 2\sin^2 A = 1 - 2\sin^2 A - 2\sin^2 A$$
$$= 1 - 4\sin^2 A:$$

$\therefore\ \sin^2 A + \frac{1}{2}\sin A = \frac{1}{4}$, which quadratic being solved by the ordinary method, gives

$$\sin A = \frac{\pm\sqrt{5} - 1}{4}:$$

and of these, the negative value is excluded by the nature of the case, because sin 18° is a positive quantity:

$$\therefore\ \sin 18^\circ = \frac{\sqrt{5} - 1}{4} = \cos 72^\circ.$$

$$\text{Hence, } \cos^2 A = 1 - \sin^2 A = 1 - \left(\frac{\sqrt{5} - 1}{4}\right)^2$$

$$= 1 - \frac{6 - 2\sqrt{5}}{16} = \frac{16 - 6 + 2\sqrt{5}}{16} = \frac{10 + 2\sqrt{5}}{16}:$$

$$\text{and } \therefore\ \cos 18^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4} = \sin 72^\circ.$$

$$\text{Also, } \cos 36^\circ = \cos 2A = \cos^2 A - \sin^2 A$$

$$= \frac{10 + 2\sqrt{5}}{16} - \frac{6 - 2\sqrt{5}}{16} = \frac{4 + 4\sqrt{5}}{16} = \frac{\sqrt{5} + 1}{4} = \sin 54^\circ:$$

$$\text{and } \sin^2 36^\circ = 1 - \cos^2 36^\circ$$

$$= 1 - \left(\frac{\sqrt{5} + 1}{4}\right)^2 = \frac{10 - 2\sqrt{5}}{16} = \cos^2 54^\circ;$$

$$\text{or, } \sin 36^\circ = \frac{\sqrt{10 - 2\sqrt{5}}}{4} = \cos 54^\circ.$$

The negative value $-\frac{1}{4}(\sqrt{5} + 1)$ of sin A, which satisfies the *Equation*, but not the *Condition of the Problem*, will be considered in the first Appendix.

52. Cor. 1. In consequence of the simplicity and similarity of the *forms* of $\sin 18^\circ = \cos 72^\circ = \frac{1}{4}(\sqrt{5} - 1)$, and of $\cos 36^\circ = \sin 54^\circ = \frac{1}{4}(\sqrt{5} + 1)$, these angles may be advantageously employed in establishing certain formulæ of verification: thus,

$$\sin(36^\circ + A) - \sin(36^\circ - A) = 2\cos 36^\circ \sin A:$$
$$\sin(72^\circ + A) - \sin(72^\circ - A) = 2\cos 72^\circ \sin A;$$

$$\therefore \sin(36^\circ + A) + \sin(72^\circ - A) - \sin(36^\circ - A) - \sin(72^\circ + A)$$
$$= 2 \sin A \{\cos 36^\circ - \cos 72^\circ\}$$
$$= 2 \sin A \{\tfrac{1}{4}(\sqrt{5} + 1) - \tfrac{1}{4}(\sqrt{5} - 1)\} = \sin A.$$

(*Euler's* Formula).

Again, $\sin(54^\circ + A) + \sin(54^\circ - A) = 2 \sin 54^\circ \cos A$:
$$\sin(18^\circ + A) + \sin(18^\circ - A) = 2 \sin 18^\circ \cos A;$$
$$\therefore \sin(54^\circ + A) + \sin(54^\circ - A) - \sin(18^\circ + A) - \sin(18^\circ - A)$$
$$= 2 \cos A \{\sin 54^\circ - \sin 18^\circ\}$$
$$= 2 \cos A \{\tfrac{1}{4}(\sqrt{5} + 1) - \tfrac{1}{4}(\sqrt{5} - 1)\} = \cos A$$

(*Legendre's* Formula).

53. Cor. 2. By substituting in the formulæ of Article (48), we shall find

$$\sin 9^\circ = \tfrac{1}{4}\{\sqrt{3 + \sqrt{5}} - \sqrt{5 - \sqrt{5}}\},$$
$$\cos 9^\circ = \tfrac{1}{4}\{\sqrt{3 + \sqrt{5}} + \sqrt{5 - \sqrt{5}}\}:$$
$$\sin 27^\circ = \tfrac{1}{4}\{\sqrt{5 + \sqrt{5}} - \sqrt{3 - \sqrt{5}}\},$$
$$\cos 27^\circ = \tfrac{1}{4}\{\sqrt{5 + \sqrt{5}} + \sqrt{3 - \sqrt{5}}\}:$$

and their values may easily be calculated, as well as additional formulæ of verification derived from them.

54. *To express the sines and cosines of two angles, in terms of the sines and cosines of their semi-sum and semi-difference.*

Since, by a well-known arithmetical theorem, the *greater* of two numbers is equal to their semi-sum *increased* by their semi-difference, and the *less* is equal to their semi-sum *diminished* by their semi-difference, we have

$$\left.\begin{aligned} A &= \tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B) = S + D \\ B &= \tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B) = S - D \end{aligned}\right\} \text{ suppose :}$$

$$\therefore \sin A = \sin(S + D) = \sin S \cos D + \cos S \sin D$$
$$= \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B) + \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B):$$
$$\sin B = \sin(S - D) = \sin S \cos D - \cos S \sin D$$
$$= \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B) - \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B):$$

similarly, $\cos A =$

$$\text{os}\ \tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) - \sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B):$$

$$\cos B =$$

$$\text{os}\ \tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) + \sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).$$

55. The chief use of the last proposition is to exress the *Sums* and *Differences* of two functions of the ame kind, in the forms of *Products;* thus, we have

$$\sin A + \sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B):$$
$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B):$$
$$\cos B + \cos A = 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B):$$
$$\cos B - \cos A = 2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B):$$

nd the order in the former two is different from that n the latter, only because cos B is greater than cos A, vhen sin A is greater than sin B, in consequence of the elation expressed by the equality

$$\sin^2 A + \cos^2 A = 1 = \sin^2 B + \cos^2 B:$$
$$\text{or, } \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A.$$

The extensive utility of these results will justify heir enunciation at full length, for the assistance of the nemory.

(1) The sum of the sines of any two angles, is qual to twice the product of the sine of their semium and the cosine of their semi-difference.

(2) The difference of the sines of any two angles, s equal to twice the product of the cosine of their semium and the sine of their semi-difference.

(3) The sum of the cosines of any two angles, is qual to twice the product of the cosine of their semium and the cosine of their semi-difference.

(4) The difference of the cosines of any two angles, s equal to twice the product of the sine of their semium and the sine of their semi-difference.

56. Cor. 1. From the last Article, by division,

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}$$

$$=\frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A+B)}\times\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A-B)}$$

$$=\tan\tfrac{1}{2}(A+B)\times\cot\tfrac{1}{2}(A-B)$$

$$=\tan\tfrac{1}{2}(A+B)\times\frac{1}{\tan\frac{1}{2}(A-B)}$$

$$=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}:$$

that is, the sum of the sines of two angles : the difference of the sines :: the tangent of their semi-sum : the tangent of their semi-difference.

Similarly, $\dfrac{\cos B+\cos A}{\cos B-\cos A}=\cot\frac{1}{2}(A+B)\cot\frac{1}{2}(A-B)$.

57. Cor. 2. From Article (55), we have

$$\frac{\sin A+\sin B}{\cos A+\cos B}=\frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}$$

$$=\frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A+B)}=\tan\tfrac{1}{2}(A+B):$$

$$\frac{\sin A-\sin B}{\cos A+\cos B}=\frac{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}$$

$$=\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A-B)}=\tan\tfrac{1}{2}(A-B):$$

both of which will hereafter be used in the establishment of important general formulæ.

58. *To express the tangents of the Sum and Difference of two angles, in terms of the tangents of the angles themselves.*

Here, from equal division of the numerator and denominator by cos A cos B, we shall have

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

$$=\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{1-\dfrac{\sin A}{\cos A}\,\dfrac{\sin B}{\cos B}}=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

Exactly in the same way; or by putting $-B$ for B, nd therefore $-\tan B$ for $\tan B$, we find

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

Since cotangent × tangent = 1, the *cotangents* of $A\pm B$ nay be immediately deduced, and expressed in terms of ot A and $\cot B$.

59. *To express tan* 2 A, *and tan* 3 A *in terms of tan* A.

Here, by means of the formula in the last Article;

$$\tan 2A=\tan(A+A)=\frac{\tan A+\tan A}{1-\tan A\tan A}=\frac{2\tan A}{1-\tan^2 A}:$$

$$\tan 3A=\tan(2A+A)=\frac{\tan 2A+\tan A}{1-\tan 2A\tan A}$$

$$=\frac{\dfrac{2\tan A}{1-\tan^2 A}+\tan A}{1-\dfrac{2\tan A}{1-\tan^2 A}\tan A}=\frac{2\tan A+\tan A-\tan^3 A}{1-\tan^2 A-2\tan^2 A}$$

$$=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}.$$ Similarly, for $\tan 4A$, &c.

60. Cor. Since $\tan 45^\circ=1$, we shall have

$$\tan(45^\circ+A)=\frac{\tan 45^\circ+\tan A}{1-\tan 45^\circ\tan A}=\frac{1+\tan A}{1-\tan A}:$$

$$\tan(45^\circ-A)=\frac{\tan 45^\circ-\tan A}{1+\tan 45^\circ\tan A}=\frac{1-\tan A}{1+\tan A}:$$

$$\therefore\ \tan(45^\circ+A)+\tan(45^\circ-A)$$

$$=\frac{1+\tan A}{1-\tan A}+\frac{1-\tan A}{1+\tan A}=\frac{2(1+\tan^2 A)}{1-\tan^2 A}=2\sec 2A:$$

and $$\tan(45^\circ+A)-\tan(45^\circ-A)=\frac{1+\tan A}{1-\tan A}-\frac{1-\tan A}{1+\tan A}$$

$$=\frac{4\tan A}{1-\tan^2 A}=2\tan 2A.$$

This last result will enable us to find the tangents of ngles greater than 45°, by means of those of angles vhich are less than 45°: thus, if $A=1^\circ$, we have

$$\tan 46^\circ=2\tan 2^\circ+\tan 44^\circ:$$

and the preceding formula when put in the forms,

$$2 \sec 2A = \tan(45^\circ + A) + \tan(45^\circ - A)$$
$$= \tan(45^\circ + A) + \cot(45^\circ + A):$$
$$\text{and } 2 \operatorname{cosec} 2A = \tan A + \cot A:$$

may be used to obtain the secants and cosecants of angles from their tangents and cotangents by addition only: thus, if $A = 1^\circ$, we have

$$2 \sec 2^\circ = \tan 46^\circ + \cot 46^\circ:$$
$$2 \operatorname{cosec} 2^\circ = \tan 1^\circ + \cot 1^\circ.$$

61. The functions of the sum of *three* or *more* angles may be expanded as in Article (54).

Thus, $\sin(A + B + C) = \sin\{(A + B) + C\}$
$= \sin(A + B) \cos C + \cos(A + B) \sin C = \sin A \cos B \cos C$
$+ \sin B \cos A \cos C + \sin C \cos A \cos B - \sin A \sin B \sin C$:

$$\cos(A + B + C) = \cos\{(A + B) + C\}$$
$= \cos(A + B) \cos C - \sin(A + B) \sin C = \cos A \cos B \cos C$
$- \cos A \sin B \sin C - \cos B \sin A \sin C - \cos C \sin A \sin B.$

Whence, by division and substitution, we have

$$\tan(A + B + C)$$
$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan A \tan C - \tan B \tan C},$$

which may also be found as in Article (59).

If A, B, C be the angles of a triangle, we have

$$\tan(A + B + C) = \tan 180^\circ = 0;$$
$$\therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C:$$

which is a remarkable property, and informs us that by fixing upon *any* two quantities, we may always find a third, so that their sum shall be equal to their continued product.

If $B = C = A$, we shall have, after reduction,

$$\sin 3A = 3 \sin A - 4 \sin^3 A, \quad \cos 3A = 4 \cos^3 A - 3 \cos A,$$

as before: from which the values of $\sin A$ and $\cos A$ may be expressed in terms of $\sin 3A$ and $\cos 3A$ respectively, by the solutions of *Cubic Equations*.

Similarly, the solution of the cubic equation,

$$\tan^3 A - 3 \tan 3A \tan^2 A - 3 \tan A + \tan 3A = 0,$$

will give the values of $\tan A$: and it thus appears that the *Trisection of an Angle* which cannot be effected by *Elementary Geometry*, is made by *Trigonometry* to depend upon the solution of a *Cubic Equation*, the roots of which, when all real, as is the case here by (7) of Article (35), are not to be ascertained by any *direct* method hitherto discovered. See *Chapter* VI.

62. The preceding Articles contain most of the formulæ of any practical use, and we will conclude this part of the subject with the demonstrations of two important general Trigonometrical Theorems, dependent upon the Principles of Algebraical Induction; observing that a variety of Examples, for the exercise of the Student, may be found at the end of the work.

63. *To find the continued product of* m *factors of the form,* $\cos a + \sqrt{-1} \sin a$.

By actual multiplication, we have

$$\{\cos \alpha + \sqrt{-1} \sin \alpha\} \{\cos \beta + \sqrt{-1} \sin \beta\}$$

$$= (\cos \alpha \cos \beta - \sin \alpha \sin \beta) + \sqrt{-1} (\sin \alpha \cos \beta + \cos \alpha \sin \beta)$$

$$= \cos (\alpha + \beta) + \sqrt{-1} \sin (\alpha + \beta) = P_2, \text{ suppose}:$$

whence, introducing another factor $\cos \gamma + \sqrt{-1} \sin \gamma$, and using the same kind of notation, we have

$$P_3 = \{\cos (\alpha + \beta) + \sqrt{-1} \sin (\alpha + \beta)\} (\cos \gamma + \sqrt{-1} \sin \gamma)$$

$$= \cos (\alpha + \beta + \gamma) + \sqrt{-1} \sin (\alpha + \beta + \gamma),$$

by the preceding: similarly, by the introduction of the factor $\cos \delta + \sqrt{-1} \sin \delta$, we have

$$P_4 = \cos (\alpha + \beta + \gamma + \delta) + \sqrt{-1} \sin (\alpha + \beta + \gamma + \delta):$$

and so on: therefore, if there be $m - 1$ factors, and

$$P_{m-1} = \cos (\alpha + \beta + \text{\&c.} + \kappa) + \sqrt{-1} \sin (\alpha + \beta + \text{\&c.} + \kappa):$$

by the introduction of the factor $\cos \lambda + \sqrt{-1} \sin \lambda$, we shall have

$$P_m = \cos (\alpha + \beta + \text{\&c.} + \lambda) + \sqrt{-1} \sin (\alpha + \beta + \text{\&c.} + \lambda).$$

that is, if the formula be true for $m-1$ factors, it is true for m factors: but it has been proved true for 2, 3, 4 *factors*, ∴ it is true for 5, 6, 7, &c. factors, and ∴ generally true.

If the algebraical signs of α, β, γ, &c. be changed, those of the *sines* are changed, but those of the *cosines* are not; and consequently the Theorem will be true, whether the symbol $\sqrt{-1}$ be affected with the positive or negative sign in both members of the equality.

If $\alpha = \beta = \gamma =$ &c. $= \lambda$ to m terms, we have

$$\{\cos \alpha \pm \sqrt{-1} \sin \alpha\}^m = \cos m\alpha \pm \sqrt{-1} \sin m\alpha,$$

which is a particular case of a formula due to *M. Demoivre*, considered by Mathematicians of an importance almost equal to that of the *Binomial Theorem* in Algebra.

64. Cor. If we effect the multiplications at length, in the latter member of the equality

$$\cos(\alpha+\beta+\gamma) + \sqrt{-1} \sin(\alpha+\beta+\gamma)$$

$$= (\cos\alpha + \sqrt{-1} \sin\alpha)(\cos\beta + \sqrt{-1} \sin\beta)(\cos\gamma + \sqrt{-1} \sin\gamma),$$

and equate the *real* and *imaginary* parts of the resulting equation respectively, we shall have, as in Article (61),

$$\cos(\alpha+\beta+\gamma) = \cos\alpha \cos\beta \cos\gamma - \cos\alpha \sin\beta \sin\gamma$$
$$- \cos\beta \sin\alpha \sin\gamma - \cos\gamma \sin\alpha \sin\beta :$$

$$\sin(\alpha+\beta+\gamma) = \sin\alpha \cos\beta \cos\gamma + \sin\beta \cos\alpha \cos\gamma$$
$$+ \sin\gamma \cos\alpha \cos\beta - \sin\alpha \sin\beta \sin\gamma.$$

The operation of Division will lead to *similar* remarkable results.

65. *To find an expression for the tangent of the sum of m angles α, β, γ, &c., λ, in terms of the tangents of the simple angles.*

Let s_1 denote the sum of the tangents:

s_2 .... of their products taken 2 & 2:

s_3 .... of their products taken 3 & 3: &c:

then will $\tan(\alpha+\beta+\gamma+\text{&c.}+\lambda) = \dfrac{s_1 - s_3 + s_5 - \text{&c.}}{1 - s_2 + s_4 - \text{&c.}}$.

For, $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{s_1}{1 - s_2}$:

$$\tan(\alpha+\beta+\gamma) = \frac{\tan(\alpha+\beta) + \tan\gamma}{1 - \tan(\alpha+\beta)\tan\gamma}$$

$$= \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - (\tan\alpha\tan\beta + \tan\alpha\tan\gamma + \tan\beta\tan\gamma)} = \frac{s_1 - s_3}{1 - s_2}:$$

and so on: wherefore if $m-1$ angles be taken, and

$$\tan(\alpha+\beta+\&c.+\kappa) = \frac{s_1 - s_3 + s_5 - \&c.}{1 - s_2 + s_4 - \&c.}:$$

then, $\tan(\alpha+\beta+\&c.+\lambda) = \tan\{(\alpha+\beta+\&c.+\kappa)+\lambda\}$

$$= \frac{\dfrac{s_1 - s_3 + s_5 - \&c.}{1 - s_2 + s_4 - \&c.} + \tan\lambda}{1 - \left(\dfrac{s_1 - s_3 + s_5 - \&c.}{1 - s_2 + s_4 - \&c.}\right)\tan\lambda}$$

$$= \frac{(s_1 - s_3 + s_5 - \&c.) + (1 - s_2 + s_4 - \&c.)\tan\lambda}{(1 - s_2 + s_4 - \&c.) - (s_1 - s_3 + s_5 - \&c.)\tan\lambda}$$

$$= \frac{(s_1 + \tan\lambda) - (s_3 + s_2\tan\lambda) + (s_5 + s_4\tan\lambda) - \&c.}{1 - (s_2 + s_1\tan\lambda) + (s_4 + s_3\tan\lambda) - \&c.}:$$

which expression for m angles is of the same *form* as that for $m-1$ angles, and therefore, as before, the Theorem is generally true.

66. COR. If m be *odd*, and of the form $4p+1$, the *last* terms of the *numerator* and *denominator* are s_m and s_{m-1} respectively; but if it be of the form $4p+3$, they are $-s_m$ and $-s_{m-1}$ respectively. If m be *even*, and of the form $4p$, the last terms of the numerator and denominator are $-s_{m-1}$ and s_m; but if of the form $4p+2$, they are s_{m-1} and $-s_m$.

Whence, by the Formulæ for Combinations, we may easily deduce the values, when α, β, γ, &c. are supposed to be equal to each other.

Ex. Let $m=5$: then $s_1 = 5\tan\alpha$:

$$s_2 = \frac{5.4}{1.2}\tan^2\alpha = 10\tan^2\alpha:$$

$$s_3 = \frac{5.4.3}{1.2.3}\tan^3\alpha = 10\tan^3\alpha: \quad s_4 = \frac{5.4.3.2}{1.2.3.4}\tan^4\alpha = 5\tan^4\alpha:$$

$$\text{and } s_5 = \frac{5.4.3.2.1}{1.2.3.4.5} \tan^5\alpha = \tan^5\alpha :$$

$$\therefore \tan 5\alpha = \frac{5 \tan\alpha - 10 \tan^3\alpha + \tan^5\alpha}{1 - 10 \tan^2\alpha + 5 \tan^4\alpha}$$

If α, β, γ be the angles of a triangle, we have $\tan(\alpha + \beta + \gamma) = 0$; and $\therefore s_1 - s_3 = 0$, or $s_1 = s_3$: that is, $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha \tan\beta \tan\gamma$, as before.

THE TRIGONOMETRICAL CANON.

67. Def. The *Trigonometrical Canon* in its simplest Form, is a *Table* or *Synopsis* of the numerical values of the sines, cosines, &c. of all angles from 1′ upwards to 90°: and it is termed a *Canon of Sines and Cosines*, when it comprises these magnitudes only.

Its Construction. The Principles and Formulæ already established will enable us to effect this in the following order.

(1) *To find the sine and cosine of* 1′.

In the formula, $\sin A = \frac{1}{2}\{\sqrt{1 + \sin 2A} - \sqrt{1 - \sin 2A}\}$, if for $2A$ we put 30°, and $\therefore \sin 2A = \frac{1}{2}$, we shall have

$$\sin \frac{30^0}{2} = .2588190 \text{ \&c.} = s_1 :$$

again, putting s_1 for $\sin 2A$, we find

$$\sin \frac{30^0}{2^2} = \cdot 1305262 \text{ \&c.} = s_2 :$$

and it is manifest that by continuing this process we shall obtain successively the sines of 15°, 7° 30′, 3° 45′, 1° 52′ 30″, &c.: also, by continued actual bisection of the angle 30°, we find at length

$$\sin \frac{30^0}{2^{10}} = .0005113269 \text{ \&c.} = s_{10} :$$

$$\sin \frac{30^0}{2^{11}} = .0002556634 \text{ \&c.} = s_{11} :$$

from which it appears that when the operation above mentioned has been repeated so many times, the sine of the angle is *halved* at the same time that the angle itself is *bisected;* that is, for *seven* or *eight* places of decimals, the sines have the same ratios as the angles:

$$\therefore \sin 1' : \sin \frac{30^\circ}{2^{11}} :: 1' : \frac{30 \times 60'}{2^{11}};$$

whence, $\sin 1' = \dfrac{2^{11}}{30 \times 60} \sin \dfrac{30^\circ}{2^{11}} = \dfrac{256}{225} \times .0002556634$ &c.

$$= .0002908882 \text{ \&c. nearly}:$$

also, from the formula $\cos A = \sqrt{1 - \sin^2 A}$, we obtain

$$\cos 1' = .999999957 \text{ \&c.}$$

COR. 1. From what has been proved in the preceding article, we have, if n be less than 60,

$$\sin n'' : \sin 60'' :: n : 60;$$

and $\therefore \sin n'' = \dfrac{n}{60} \sin 1'$, may be determined, and consequently $\cos n''$ as above.

Hence, $\sin 10'' = \frac{1}{6} \sin 1' = \frac{1}{6} (.0002908882 \text{ \&c.})$

$$= .0000484813 \text{ \&c.}$$

$$\text{and } \therefore \cos 10'' = .9999999988 \text{ \&c}:$$

and it will not fail to be perceived that $\sin 10''$ = *ten times* the circular measure of 1″, as given in Article (15).

COR. 2. This process, though laborious, will enable us to find an approximate value of the symbol π before used: thus, π contains $180^\circ = 180 \times 60$ minutes

$$= 3 \times 60 \times 60 \text{ minutes}:$$

$$\begin{aligned} \therefore \text{ we have } \pi &= .0002908882 \text{ \&c.} \times 3 \times 60 \times 60 \\ &= .0008726646 \text{ \&c.} \times 60 \times 60 \\ &= .052359876 \text{ \&c.} \times 60 \\ &= 3.14159256 \text{ \&c.}: \end{aligned}$$

which is the numerical value of the *semi-circumference* of a circle whose *radius* is 1, and therefore of the *whole circumference* of a circle whose *diameter* is 1: and it is quite accurate to *six* places of decimals.

(2) *To find the sines and cosines of angles from* 1′ *to* 1°.

Since, $\sin (A + B) = 2 \sin A \cos B - \sin (A - B)$

$$\begin{aligned} &= 2 \sin A \{1 - 2 \sin^2 \tfrac{1}{2} B\} - \sin (A - B) \\ &= 2 \sin A - \sin (A - B) - 4 \sin A \sin^2 \tfrac{1}{2} B: \end{aligned}$$

if we suppose $B = 1'$, and A to take the values $1'$, $2'$, $3'$, &c. in succession, we have

$$\sin 2' = \sin 1' + (\sin 1' - \sin 0') - 4 \sin 1' \sin^2 30'' :$$

$$\sin 3' = \sin 2' + (\sin 2' - \sin 1') - 4 \sin 2' \sin^2 30'' : \text{ \&c.} :$$

$$\text{where, } \sin 30'' = \frac{30}{60} \sin 1' = \tfrac{1}{2} \sin 1'$$

is readily found: and the *form* in which the quantities are here put will facilitate the calculation.

The same might have been done by means of the formula, $\sin (m+1)A = 2 \cos A \sin mA - \sin (m-1)A$; and the corresponding formula, $\cos (m+1)A = 2 \cos A \cos mA - \cos (m-1)A$, will give the values of the cosines.

Thus, $\sin (m+1)10'' = 2 \cos 10'' \sin m10'' - \sin (m-1)10''$,

$$\cos (m+1)10'' = 2 \cos 10'' \cos m10'' - \cos (m-1)10'' :$$

and since $2 \cos 10'' = 1.9999999976$ &c.

$$= 2 - .0000000024 \text{ \&c.}$$

we shall have

$$\sin (m+1)10''$$

$$= (2 - .0000000024 \text{ \&c.}) \sin m10'' - \sin (m-1)10''$$

$$= 2 \sin m10'' - \sin (m-1)10'' - .0000000024 \text{ \&c. } \sin m10'' :$$

$$\cos (m+1)10''$$

$$= 2 \cos m10'' - \cos (m-1)10'' - .0000000024 \text{ \&c. } \cos m10'' :$$

and thus the sines and cosines of angles differing by 10″ may be derived from those immediately preceding by subtraction, after the multiplication required only in the last terms.

Hence, also, $\sin (m+1)10'' - \sin m10''$

$$= \sin m10'' - \sin (m-1)10'' - .0000000024 \text{ \&c. } \sin m10'',$$

which furnishes a rule for determining the diminished difference of the sines of consecutive angles increasing by 10″. Similarly, for the cosines.

This process may evidently be continued for angles of all magnitudes, if it be found necessary.

(3) *To find the sines and cosines of angles from* 1^0 *to* 30^0.

The formulæ which may be used are,

$$\sin(A+B) = \frac{(\sin A + \sin B)(\sin A - \sin B)}{\sin(A-B)} :$$

$$\cos(A+B) = \frac{(\cos A + \sin B)(\cos A - \sin B)}{\cos(A-B)} :$$

$$\text{thus,}\ \sin 3^\circ = \frac{(\sin 2^\circ + \sin 1^\circ)(\sin 2^\circ - \sin 1^\circ)}{\sin 1^\circ} : \ \text{\&c.}$$

$$\cos 3^\circ = \frac{(\cos 2^\circ + \sin 1^\circ)(\cos 2^\circ - \sin 1^\circ)}{\cos 1^\circ} : \ \text{\&c.}$$

he values of $\sin 2^\circ$ and $\cos 2^\circ$, being first computed by eans of the formulæ

$$\sin 2A = 2 \sin A \cos A, \text{ and } \cos 2A = 2\cos^2 A - 1.$$

(4) *To find the sines and cosines of angles from* 0° *to* 45°.

Since, $\sin(30^\circ + A) = \cos A - \sin(30^\circ - A)$:

$\cos(30^\circ + A) = \cos(30^\circ - A) - \sin A$:

f we suppose $A = 1', 2', 3'$, &c. we shall have

$\sin 30^\circ 1' = \cos 1' - \sin 29^\circ 59'$: &c.

$\cos 30^\circ 1' = \cos 29^\circ 59' - \sin 1'$: &c.

vhich may be continued as far as we please: and if $A = 10''$, $20''$, &c. the same formulæ may be applied up o 45°.

(5) *To find the sines and cosines of angles from* 5° *to* 90°, *and upwards.*

This is done immediately by means of the formulæ

$$\sin(45^\circ + A) = \cos(45^\circ - A) :$$
$$\cos(45^\circ + A) = \sin(45^\circ - A).$$

lso, $\sin(90^\circ + A) = + \cos A$: $\cos(90^\circ + A) = - \sin A$:

$\sin(180^\circ + A) = - \sin A$: $\cos(180^\circ + A) = - \cos A$:

$\sin(270^\circ + A) = - \cos A$: $\cos(270^\circ + A) = + \sin A$:

ut these latter functions are never put down in the ables: and indeed a table extending only to 45°, is quite ufficient for every angle, as is already manifest.

(6) A table of *tangents* and *cotangents*, may be readily constructed by the use of the fundamental formulæ:

$$\tan A = \frac{\sin A}{\cos A}; \ \cot A = \frac{\cos A}{\sin A}:$$

and the operations will be abridged by means of

$$\tan (45^\circ + A) = \tan (45^\circ - A) + 2 \tan 2A:$$
$$\cot (45^\circ + A) = \cot (45^\circ - A) - 2 \tan 2A.$$

(7) The *secants* and *cosecants* are not always found in the tables, but they may be computed immediately from the formulæ:

$$\sec A = \frac{1}{\cos A}; \ \operatorname{cosec} A = \frac{1}{\sin A}:$$

and they may be found by Addition only from the formulæ established in Article (60): also, the *Versed sines* are had from $\operatorname{vers} A = 1 - \cos A$.

What has here been given, is merely an outline of the plan which may be adopted, without reference to higher principles: and indeed the more scientific analytical methods could not properly be introduced in this place, though some of them will be seen hereafter.

Its Verification. As a check upon the computations already noticed, *Formulæ of Verification* are made use of, in order to detect any error which may have been entailed upon them, by reason of the *Method of Continuation* which we have pursued. Thus,

$$\sin 3^\circ = \sin (18^\circ - 15^\circ), \text{ and } \cos 3^\circ = \cos (18^\circ - 15^\circ),$$

are readily computed to as much nicety as is necessary: whence, sin 6° and cos 6° are obtained by Article (44): also, the sines and cosines of the series of angles 9°, 12°, &c., whose common difference is 3°, may be found by the same *independent methods*, and when arrived at by the *process of continuation* explained in the preceding articles, will certify us of the accuracy of the arithmetical operations which have been performed, unless a *compensation of errors* has taken place.

Formulæ of this description may be made as numerous as we please, but those in greatest use have already been mentioned, and are

1) $$\text{Sin}\,\tfrac{1}{2}A = \tfrac{1}{2}\left\{\sqrt{1+\sin A} - \sqrt{1-\sin A}\right\},$$
$$\cos\tfrac{1}{2}A = \tfrac{1}{2}\left\{\sqrt{1+\sin A} + \sqrt{1-\sin A}\right\},$$

se application may be seen in Article (48).

2) $$\text{Sin}\,A = \sin(36^\circ + A) - \sin(36^\circ - A)$$
$$- \sin(72^\circ + A) + \sin(72^\circ - A):$$
$$\cos A = \sin(54^\circ + A) + \sin(54^\circ - A)$$
$$- \sin(18^\circ + A) - \sin(18^\circ - A):$$

in a table of sines and cosines correctly computed n $A = 5^\circ$ and 7°, we must have respectively,

$$\sin 5^\circ = \sin 41^\circ + \sin 67^\circ - \sin 31^\circ - \sin 77^\circ,$$
$$\cos 7^\circ = \sin 61^\circ + \sin 47^\circ - \sin 25^\circ - \sin 11^\circ:$$

therefore, if these equalities be not maintained, we r that an error has been committed in the computa- of one or more of the functions employed.

3) $$\text{Tan}\,(45^\circ + A) = \tan(45^\circ - A) + 2\tan 2A:$$
$$\tan(45^\circ + A) = 2\sec 2A - \tan(45^\circ - A).$$

The Arithmetical values of the trigonometrical ratios considered, are called the ***natural*** sines, cosines, &c.: by Article (34) they may be adapted to the ***Tabular*** ***ius*** 10^{10}, which is requisite for some purposes.

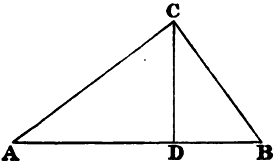
CHAPTER III.

THE PROPERTIES OF PLANE TRIANGLES, AND REGULAR POLYGONS. THE CIRCUMFERENCE AND AREA OF A CIRCLE.

68. **Def.** In every plane triangle ABC, there are *six* distinct parts: the *three* angles A, B, C and the *three* sides BC, AC, AB respectively opposite to them, which are usually denoted by the corresponding small letters a, b, c: and our first object will be to express in terms of the latter, the functions of the former as defined in Article (24), and the contrary: and thus, by means of their mutual dependence on each other, to investigate commodious formulæ for the *Solution of Triangles*, which will be explained and applied in the next chapter.

69. *The sines of the angles of a triangle have the same ratios to each other, as the sides by which they are respectively subtended.*

Let ABC be a triangle whose angles and sides are A, B, C; a, b, c respectively: from any angle C draw CD perpendicular to AB, produced, if necessary:



then, we have $\sin A = \dfrac{CD}{AC}$; $\sin B = \dfrac{CD}{BC}$:

$$\therefore\ \sin A : \sin B = \frac{CD}{AC} : \frac{CD}{BC} = \frac{1}{AC} : \frac{1}{BC}$$

$$= BC : AC = a : b;$$

similarly, by drawing from A a perpendicular upon BC, produced, if necessary, we shall have

$$\sin B : \sin C = b : c;$$

and therefore by the composition of ratios, we obtain

$$\sin A \;:\; \sin C = a \;:\; c;$$

that is, $\dfrac{\sin A}{\sin B} = \dfrac{a}{b}$, $\dfrac{\sin B}{\sin C} = \dfrac{b}{c}$, $\dfrac{\sin A}{\sin C} = \dfrac{a}{c}$:

which are sometimes expressed in the form,

$$\sin A \;:\; \sin B \;:\; \sin C = a \;:\; b \;:\; c.$$

70. Cor. Since $a : b = \sin A : \sin B$;

we have $a + b \;:\; a - b = \sin A + \sin B \;:\; \sin A - \sin B$

$$= 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B) : 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B)$$

$$= \frac{\sin \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A + B)} : \frac{\sin \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A - B)}$$

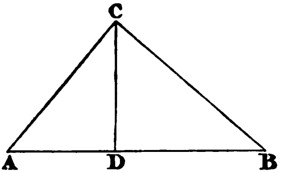
$$= \tan \tfrac{1}{2}(A + B) \;:\; \tan \tfrac{1}{2}(A - B);$$

hat is, the sum of any two sides : their difference :: the angent of half the sum of their opposite angles : the angent of half the difference.

71. *To express the cosine of an angle of a triangle, n terms of the sides.*

Retaining the notation and construction above used, we shall have the two following cases:

(1) *When the angle is acute.*



Let $AD = x$, and therefore $DB = c - x$: then will

$$\cos A = \frac{AD}{AC} = \frac{x}{b}, \text{ or, } b \cos A = x:$$

but $x^2 = AD^2 = AC^2 - CD^2 = AC^2 - BC^2 + DB^2$

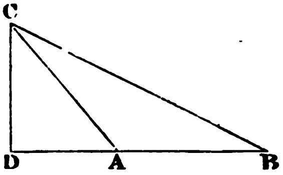
$$= b^2 - a^2 + (c - x)^2$$

$$= b^2 - a^2 + c^2 - 2cx + x^2;$$

$\therefore\; 2cx = b^2 + c^2 - a^2$: and $2bc \cos A = b^2 + c^2 - a^2$;

whence, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

(2) *When the angle is obtuse.*



Let $AD = x$, and therefore $DB = c + x$: then will

$$\cos A = \frac{AD}{AC} = \frac{-x}{b}, \text{ or, } b \cos A = -x :$$

$$\text{but } x^2 = AD^2 = AC^2 - CD^2 = AC^2 - BC^2 + DB^2$$
$$= b^2 - a^2 + (c + x)^2$$
$$= b^2 - a^2 + c^2 + 2cx + x^2 ;$$

$$\therefore -2cx = b^2 + c^2 - a^2 : \text{ and } 2bc \cos A = b^2 + c^2 - a^2 ;$$

$$\text{whence, } \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

It appears therefore, that the cosine of an angle of a triangle, whether *acute* or *obtuse*, is expressed in the same *form*, though it is *positive* in the former case, and *negative* in the latter.

The values of the cosines of B and C will evidently be

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}, \text{ and } \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

These are merely the results of propositions (12) and (13) of *Euclid* II, expressed by Trigonometry; and the *form* of the equality $a^2 = b^2 + c^2 - 2bc \cos A$, which determines any side of a triangle by means of the two others and their included angle, it will be worth while to recollect.

72. Cor. Since $2bc \cos A = b^2 + c^2 - a^2$, we have

$$a^2 - b^2, \text{ or } (a + b)(a - b) = c(c - 2b \cos A) ;$$

$$\text{and } \therefore c : a + b = a - b : c - 2b \cos A :$$

$$\text{but when } A \text{ is acute, } c - 2b \cos A = AB - 2AD$$
$$= (AB - AD) - AD = BD - AD :$$

and when A is obtuse, $c - 2b\cos A = AB + 2AD$

$$= (AB + AD) + AD = BD + AD:$$

∴ expressing them in geometrical symbols, we have

$$AB : BC + AC :: BC - AC : BD \mp AD;$$

or, the base : the sum of the other sides :: their difference : the *difference* or *sum* of the lines intercepted between the extremities of the base and the perpendicular, according as it falls *within* or *without* the triangle.

This proportion is generally used by Practical Mathematicians not much accustomed to the treatment of Algebraical Formulæ.

73. *To express the sine of an angle of a triangle, in terms of the sides.*

Here, $\sin^2 A = 1 - \cos^2 A = (1 - \cos A)(1 + \cos A)$:

$$\text{but, } 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{a^2 - (b^2 - 2bc + c^2)}{2bc} = \frac{a^2 - (b - c)^2}{2bc} = \frac{(a + b - c)(a - b + c)}{2bc}:$$

$$\text{and } 1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc + b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(b^2 + 2bc + c^2) - a^2}{2bc} = \frac{(b + c)^2 - a^2}{2bc} = \frac{(b + c + a)(b + c - a)}{2bc}:$$

∴ for the sake of conciseness putting $2s = a + b + c$;

we have, $2(s - a) = a + b + c - 2a = b + c - a$:

$$2(s - b) = a + b + c - 2b = a + c - b:$$

$$2(s - c) = a + b + c - 2c = a + b - c:$$

$$\therefore\ 1 - \cos A = \frac{2(s - b)\,2(s - c)}{2bc}:\quad 1 + \cos A = \frac{2s\,2(s - a)}{2bc}:$$

$$\text{that is, } \sin^2 A = \frac{4}{b^2c^2}\, s(s - a)(s - b)(s - c):$$

$$\text{or, } \sin A = \frac{2}{bc}\sqrt{s(s - a)(s - b)(s - c)}.$$

74. Cor. If $\sqrt{s(s - a)(s - b)(s - c)}$, which is *symmetrical* with respect to the three sides a, b, c, be denoted by S; we shall have

$$\sin A = \frac{2S}{bc},\ \sin B = \frac{2S}{ac},\ \sin C = \frac{2S}{ab}:$$

$$\text{and } \therefore \sin A : \sin B : \sin C = \frac{2S}{bc} : \frac{2S}{ac} : \frac{2S}{ab}$$

$$= a : b : c;$$

and this is the result of Article (69), established by a less simple, though *independent* process: inasmuch as no use has been made of it in Articles (71) and (73).

75. *To express the sine, cosine, and tangent of half an angle of a triangle, in terms of the sides.*

Following the steps of the process of the last article but one, and using the formulæ of Article (45), we have

$$2\sin^2 \tfrac{1}{2}A = 1 - \cos A = \frac{2(s-b)\,2(s-c)}{2bc}:$$

$$\text{and } \therefore \sin \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}:$$

$$2\cos^2 \tfrac{1}{2}A = 1 + \cos A = \frac{2s\,2(s-a)}{2bc}:$$

$$\text{and } \therefore \cos \tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}:$$

$$\text{whence, } \tan \tfrac{1}{2}A = \frac{\sin \frac{1}{2}A}{\cos \frac{1}{2}A} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}:$$

and the algebraical signs of all these quantities must be positive, since $\frac{1}{2}A$ is necessarily less than 90°.

The same functions of the other angles will be obtained immediately, by the proper changes of symbols.

If $\sin \frac{1}{2}A$ and $\cos \frac{1}{2}A$, be already found, $\sin A$ may be had from the equality

$$\sin A = 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A.$$

76. *To express the Area of a triangle, in terms of the sides.*

Since the area of any triangle is *half* that of a rectangular parallelogram of the same base and altitude, by *Euclid*, Prop. 41. Book I: we have, from Article (192) *of the Arithmetic:*

$$\text{the area of the triangle } ABC = \tfrac{1}{2}AB \times CD$$

$$= \tfrac{1}{2}AB \, . \, AC \, . \, \frac{CD}{AC}$$

$$= \tfrac{1}{2}AB \, . \, AC \, . \, \sin A = \frac{bc}{2} \, . \, \frac{2}{bc} \sqrt{s(s-a)\,(s-b)\,(s-c)}$$

$$= \sqrt{s(s-a)\,(s-b)\,(s-c)}.$$

Hence, the Rule enunciated at length will be:

From half the sum of the sides, subtract each side separately; multiply the half-sum and the three remainders together, and the square root of the product will be the area.

Ex. Let it be required to find the area of the triangle, whose three sides are 18, 24, and 30 yards.

Here, $2s = 18 + 24 + 30 = 72$: $\therefore\ s = 36$:

$s-a = 36-18 = 18$, $s-b = 36-24 = 12$, $s-c = 36-30 = 6$:

whence, the area $= \sqrt{36 \, . \, 18 \, . \, 12 \, . \, 6} = \sqrt{36 \, . \, 9 \, . \, 2 \, . \, 2 \, . \, 6 \, . \, 6}$

$$= \sqrt{6^2 . \, 3^2 . \, 2^2 . \, 6^2} = 6 \, . \, 3 \, . \, 2 \, . \, 6 = 216:$$

that is, if the three sides be 18, 24 and 30 *lineal* yards, the area will be 216 *superficial* or *square* yards.

Also, the area $= \frac{1}{4}\sqrt{2\,(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)}$, which will be practically convenient when a, b, c are *irrational quantities*.

77. Cor. 1. The symbol S adopted in Article (74), will therefore represent the area of the triangle ABC: and we have from Article (76);

$$S = \tfrac{1}{2}\, AB \, . \, AC \sin A:$$

or, the area of a triangle may be *arithmetically* obtained by multiplying half the product of any two sides by the sine of the angle included between them.

It is also evident that the *Area* of the triangle may always be used as an *Element* of calculation, whenever its introduction will simplify the process.

78. Cor. 2. The *Segments* of the base, and the *Perpendicular* upon it are easily found: thus,

$$AD = \frac{b^2 + c^2 - a^2}{2c} \; : \; BD = \frac{a^2 + c^2 - b^2}{2c} :$$

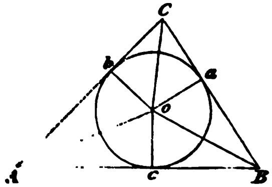
$$CD = \frac{2}{c} \sqrt{s(s-a)(s-b)(s-c)}.$$

The area, and each of these lines, are readily deducible from *Geometrical Principles*, and may be expressed in a variety of different forms by substitution.

79. *Of a triangle, to express the radii of the inscribed and circumscribed Circles, in terms of the sides.*

(1) *The Inscribed Circle.*

If o be the centre of the circle inscribed in the triangle ABC, we have $oa = ob = oc =$ the radius r:



$$\text{and } S = \triangle AoB + \triangle AoC + \triangle BoC$$

$$= \tfrac{1}{2} cr + \tfrac{1}{2} br + \tfrac{1}{2} ar = \frac{a+b+c}{2} r = rs :$$

$$\therefore r = \frac{S}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}.$$

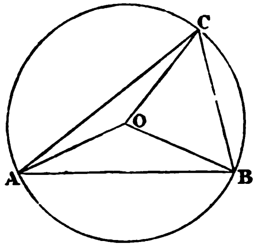
Hence, $AoB = 180° - \tfrac{1}{2}A - \tfrac{1}{2}B = 90° + \tfrac{1}{2}C$, and we have

$$Ao : AB = \sin \tfrac{1}{2}B : \cos \tfrac{1}{2}C = \sqrt{b(s-a)} : \sqrt{cs};$$

$$\text{which gives } Ao = AB \sqrt{\frac{b(s-a)}{cs}} = \frac{2bc}{a+b+c} \cos \tfrac{1}{2}A.$$

(2) *The Circumscribed Circle.*

If O be the centre of the circle circumscribing the triangle ABC, we have $OA = OB = OC =$ the radius R: then since the angle at the centre of a circle is double of the angle at the circumference upon the same base, we have, from Article (77),



$$S = \tfrac{1}{2}ab \sin \tfrac{1}{2}AOB: \text{ and } \sin \tfrac{1}{2}AOB = \frac{\tfrac{1}{2}AB}{AO} = \frac{c}{2R}:$$

$$\therefore\ S = \frac{abc}{4R}: \text{ or, } R = \frac{abc}{4S} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

From this, it appears that the sine of any angle of a riangle is equal to the *ratio* of the opposite side to the iameter of the circumscribed circle.

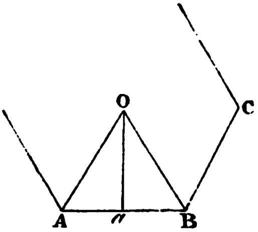
80. Cor. Hence it immediately follows that

$$Rr = \frac{abc}{4S} \cdot \frac{S}{s} = \frac{abc}{4s}; \text{ and } \therefore\ 2Rr = \frac{abc}{2s} = \frac{abc}{a+b+c}.$$

Also, by means of Article (78), we have the equality, $R \cdot CD = ab = AC \cdot BC$, if CD be perpendicular to AB: nd this is Proposition C of the *sixth* Book of *Simson's* Edition of *Euclid's* Elements of Geometry.

81. *To express the Area of a regular polygon in erms of the side: and also, the radii of the inscribed nd circumscribed Circles.*

Let AB, BC be two adjacent sides of the polygon ach $= a$: bisect the angles at A and B by the straight nes



AO, BO meeting in O: *then, if* n *be the number of sides of the polygon,* we have

$AOB = \frac{360^\circ}{n}$, and $\therefore AOa = \frac{180^\circ}{n}$: also, if S be the area,

$$S = n \triangle AOB = nAa . Oa = nAa . Aa \cot\frac{180^\circ}{n} = \tfrac{1}{4}na^2 \cot\frac{180^\circ}{n}.$$

$$\text{Also, } r = Oa = Aa \cot AOa = \tfrac{1}{2}a \cot\frac{180^\circ}{n}:$$

$$\text{and } R = OA = Aa \operatorname{cosec} AOa = \tfrac{1}{2}a \operatorname{cosec}\frac{180^\circ}{n}.$$

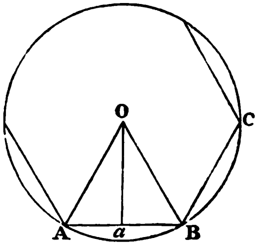
$$\text{Whence, } \frac{r}{R} = \tfrac{1}{2}a \cot\frac{180^\circ}{n} \div \tfrac{1}{2}a \operatorname{cosec}\frac{180^\circ}{n} = \cos\frac{180^\circ}{n}.$$

By assigning to n, the values 3, 4, &c. we shall have for an *Equilateral Triangle, Square,* &c. respectively,

$$\frac{r}{R} = \cos 60^\circ = \tfrac{1}{2} \ : \ \frac{r}{R} = \cos 45^\circ = \frac{1}{\sqrt{2}} : \ \&c.$$

82. *To find the Perimeters and Areas of the regular polygons of a given number of sides, inscribed in, and circumscribed about, a given circle.*

(1) *The Inscribed Polygon.*



Let r be the radius of the circle, n the number of sides of the polygon: then, the angle AOB subtended at the centre of the circle by each side of the polygon $= \frac{360^\circ}{n}$:

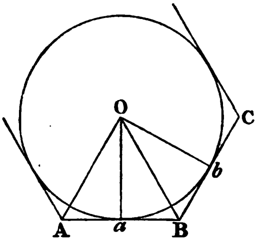
$$\text{and } AB = 2Aa = 2OA \sin AOa = 2r \sin\frac{180^\circ}{n}:$$

$$\therefore \text{ the perimeter of the polygon} = 2nr \sin\frac{180^\circ}{n}.$$

Also, $\triangle AOB = \tfrac{1}{2} OA \,.\, OB \sin AOB = \tfrac{1}{2} r^2 \sin \frac{360^\circ}{n}$:

$$\therefore \text{ the area of the polygon} = \tfrac{1}{2} nr^2 \sin \frac{360^\circ}{n}.$$

(2) *The Circumscribed Polygon.*



Here, $AB = 2Aa = 2Oa \tan AOa = 2r \tan \frac{180^\circ}{n}$:

$$\therefore \text{ the perimeter of the polygon} = 2nr \tan \frac{180^\circ}{n}.$$

Also, $\triangle AOB = Oa \,.\, Aa = Oa \,.\, Oa \tan AOa = r^2 \tan \frac{180^\circ}{n}$:

$$\therefore \text{ the area of the polygon} = nr^2 \tan \frac{180^\circ}{n}.$$

By giving the proper values to n, the perimeters and reas of all regular polygons are easily *computed* to ny degree of nicety for practical purposes.

83. Cor. If p, P denote the perimeters; and a, A he areas of the inscribed and circumscribed polygons espectively, we have

$$\frac{p}{P} = \cos \frac{180^\circ}{n}, \text{ and } \frac{a}{A} = \cos^2 \frac{180^\circ}{n}:$$

and thus, the perimeters and areas of the inscribed and circumscribed polygons may be *compared* in all cases, by assigning the proper integral values to n.

84. *To express the Circumference and Area of a circle, n terms of the radius.*

We have just seen that $\frac{p}{P} = \cos \frac{180^\circ}{n}$, $\frac{a}{A} = \cos^2 \frac{180^\circ}{n}$: whence, if the value of n be supposed to be indefinitely *increased*, that of $\frac{180^\circ}{n}$ will be indefinitely *diminished*: and in this case, $\cos\frac{180^\circ}{n}$ *continually* approaches to 1:

$$\therefore \frac{p}{P}=1, \text{ or } p=P; \text{ and } \frac{a}{A}=1, \text{ or } a=A:$$

that is, the perimeters and areas of the inscribed and circumscribed polygons are *ultimately* equal to each other, and the circle *coincides* with each of them: but since $\frac{1}{n}$th part of the *perimeter* of either *polygon* has become identical with $\frac{1}{n}$th part of the *periphery* of the *circle*, we must now have $2r\sin\frac{180^\circ}{n}$ and $2r\tan\frac{180^\circ}{n}$, each equal to $\frac{2\pi r}{n}$, if 2π denote the ratio of the circumference of a circle to its radius: and thence arise the following equalities;

$$p=2nr\sin\frac{180^\circ}{n}=2\pi r,\quad a=\tfrac{1}{2}nr^2\sin\frac{360^\circ}{n}=\pi r^2:$$

$$P=2nr\tan\frac{180^\circ}{n}=2\pi r,\quad A=nr^2\tan\frac{180^\circ}{n}=\pi r^2:$$

that is, when π denotes the *numerical magnitude* 3.1415926 &c. found in Article (67), and r represents the *number* of units contained in the radius, the *numerical* value of the *circumference* will be $2\pi r$, and that of the *area* or *surface* will be πr^2.

85. Cor. 1. Hence, the area of a circular sector is easily found.

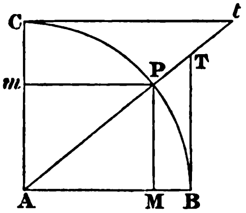
For, if a be the arc of the sector, and r its radius, we have, by *Euclid*, Prop. 33. Book VI.:
the area of the sector : the area of the circle :: a : $2\pi r$;

$$\therefore \text{ the area of the sector } = \frac{a}{2\pi r}\times\text{ the area of the circle}$$

$$=\frac{a}{2\pi r}\times\pi r^2=\tfrac{1}{2}ar:$$

or, the area of a circular sector is equal to half the rectangle contained by the radius of the circle, and a straight line equal to the arc.

86. Cor. 2. In the following diagram, we see that if



a = the arc BP, we shall have $\sin a = MP$, and $\tan a = BT$, when the circular measure is used, by Article (20): now, the sector ABP is always *intermediate* in magnitude to the triangles ABP and ABT, if BP be drawn:

$$\text{that is, } \tfrac{1}{2}AB\,.\,BP > \tfrac{1}{2}AB\,.\,MP < \tfrac{1}{2}AB\,.\,BT,$$

for *all* conditions of the quantities concerned:

whence, BP is *always* $> MP$ and $< BT$:

and therefore, the numerical magnitude of a, when the angle BAP is *acute*, is necessarily *limited* by those of $\sin a$ and $\tan a$; but because $\frac{\sin a}{\tan a} = \cos a = \cos 0^\circ = 1$, *ultimately*, by Article (30), we are led to infer that the arc, sine, and tangent are *equal* to one another, when they are *too small* to be appreciated by the powers of the mind: and though they cannot *themselves* be subjected to computation, their *relations* are thus perfectly defined.

The results which we have here established, that $\frac{a}{\sin a} = \frac{a}{\tan a} = 1$, in the *ultimate* state of a, are frequently assumed in works of science, but are not easily entered into by the mere arithmetician.

87. From Article (84), it appears that the circumferences of circles are proportional to their radii, since $p = 2\pi r$, and π is an invariable magnitude: the same conclusion may, however, be drawn from other considerations.

Let regular polygons of the same number of sides be inscribed in two circles whose radii are r_1 and r_2: then, if c_1 and c_2 be the circumferences of these circles, p_1 and p_2 the *perimeters of* their inscribed polygons, and we assume $x_1 = c_1 - p_1$, $x_2 = c_2 - p_2$: from Article (82) we shall have

$$\frac{r_1}{r_2} = \frac{p_1}{p_2} = \frac{c_1 - x_1}{c_2 - x_2}:$$

$$\text{and } \therefore c_2 r_1 - r_1 x_2 = c_1 r_2 - r_2 x_1:$$

wherein $r_1 x_2$ and $r_2 x_1$ *alone*, undergo any change from altering the number of sides of the polygons; in other words, wherein $c_2 r_1$ and $c_1 r_2$ are *invariable*, but $r_1 x_2$ and $r_2 x_1$ are *variable:* now, if an equation subsist between two sets of quantities, one part of each being variable and the other part invariable, it follows that the variable and invariable parts in each must be independent of each other, or, that the invariable parts in each member of the equality must be equal to each other:

$$\text{hence, we have } c_2 r_1 = c_1 r_2: \text{ and } \therefore \frac{c_1}{c_2} = \frac{r_1}{r_2};$$

or, the circumferences of circles are proportional to their radii.

In the same way, the areas of circles are proved to be proportional to the squares of their radii, because the areas of similar polygons inscribed in them are in that ratio.

CHAPTER IV.

THE NATURE AND PROPERTIES OF LOGARITHMS. THE ADAPTATION OF TRIGONOMETRICAL FORMULÆ TO LOGARITHMIC COMPUTATION, AND THE USE OF SUBSIDIARY ANGLES. THE SOLUTION OF TRIANGLES.

I. *The Nature of Logarithms.*

88. Def. The *Logarithm* of a number y, is such a value of the index x, of a fixed magnitude a, as will satisfy the equation $y = a^x$: that is, x is *defined* to be the logarithm of y in a *System of Logarithms* whose *Base* is a; and the logarithm of y will therefore depend entirely upon the quantity a, which may be assumed of any magnitude whatever, *unity* only excepted, because every arithmetical power or root of 1 is 1. See Chapter VII.

89. Cor. Since $1 = a^0$, we have 0 for the logarithm of 1 in every system of logarithms.

Also, because $a = a^1$, it follows that the logarithm of the base of the system is always $= 1$.

Again, since $0 = \frac{1}{a^\infty} = a^{-\infty}$, it appears that the logarithm of 0 is an infinite *negative* quantity.

And, because the equation $-y = a^x$, or $y = -a^x$, cannot be satisfied by any value of x, whether positive or negative, the logarithms of negative quantities in a system whose base is a positive arithmetical magnitude, can have no existence, or are *imaginary*.

90. If the number 10, which is the base of the common system of notation, be adopted for the base of the logarithms as above defined, and the word *logarithm* be written in the abbreviated form *log*: we have

$1 = 10^0$, or $\log 1 = 0$:
$10 = 10^1$, or $\log 10 = 1$:
$100 = 10^2$, or $\log 100 = 2$:
$1000 = 10^3$, or $\log 1000 = 3$:
$10000 = 10^4$, or $\log 10000 = 4$:
$100000 = 10^5$, or $\log 100000 = 5$: &c.

whence it is easily seen that the logarithm of any magnitude between 1 and 10, will be a fraction, which is usually expressed *decimally:* that of any magnitude between 10 and 100 will be 1, with a decimal fraction annexed: that of one between 100 and 1000 will be 2, with a corresponding decimal fraction; and so on.

Thus, in the following instances wherein the operations are easily performed, we have .

$$10^{\frac{1}{2}} = 10^{.5} = 3.16227 \text{ \&c.} : \therefore \log 3.16227 \text{ \&c.} = 0.5:$$

$$10^{\frac{3}{4}} = 10^{.75} = 5.62341 \text{ \&c.} : \therefore \log 5.62341 \text{ \&c.} = 0.75:$$

$$10^{\frac{3}{2}} = 10^{1.5} = 31.62277 \text{ \&c.} : \therefore \log 31.62277 \text{ \&c.} = 1.5 : \text{\&c.}$$

91. Def. The integers 0, 1, 2, 3, &c. to the *left* of the decimal points in the logarithms of magnitudes are called the *Characteristics* of those logarithms; and the fractional parts expressed decimally are termed their *Mantissæ*, or over-weights.

92. Cor. If the values of x in the equation $y = 10^x$ calculated for successive numerical magnitudes of y from 1 upwards, by methods hereafter to be explained, be *registered*, or put in the form of a *Table*, these values are called the *Tabular* logarithms of the corresponding numbers in the *Common* System of logarithms: which is generally called *Briggs's* System, inasmuch as *Mr. Henry Briggs*, Professor of Geometry at *Oxford*, first suggested the adoption of 10 as the most convenient number for the base of a system.

In *Babbage's* tables, which are the most correct, and by far the best adapted to practical purposes, the mantissæ alone of all numbers from 1 to 108000, calculated to *seven* places of decimals, are inserted: and the characteristics are immediately supplied from the considerations introduced in Article (90).

II. *The Properties of Logarithms.*

93. Def. The principal object of the Invention of Logarithms is to facilitate the performance of the Arithmetical Operations of *Multiplication, Division, Involution, and Evolution;* more particularly in cases where the *quantities* employed consist of several figures, and near

approximations to the true results are considered sufficient for the practical purposes to which they are applied: how this is done will now be explained and exemplified.

(1) $$\text{Log } PQ = \log P + \log Q.$$

For, let $p = \log P$, and $q = \log Q$, in a system of logarithms whose base is a: then, by the definition given in Article (88), we have

$$P = a^p, \text{ and } Q = a^q:$$

$$\text{whence, } PQ = a^p \times a^q = a^{p+q}:$$

$$\text{and } \therefore \log PQ = p + q = \log P + \log Q.$$

$$\text{Similarly, } \log PQRS = \log PQ + \log RS$$

$$= \log P + \log Q + \log R + \log S: \text{ and so on:}$$

that is, the logarithm of a product, or of a composite number, is equal to the sum of the logarithms of all its factors, and *vice versâ*.

Thus, with the help of tables, the operation of *Multiplication* is reduced to that of *Addition:* and the logarithms of *composite* numbers are derived from those of their factors.

(2) $$\text{Log } \frac{P}{Q} = \log P - \log Q.$$

For, retaining the same notation, we have

$$\frac{P}{Q} = \frac{a^p}{a^q} = a^{p-q}:$$

$$\text{and } \therefore \log \frac{P}{Q} = p - q = \log P - \log Q.$$

$$\text{Similarly, } \log \frac{PQ}{RS} = \log PQ - \log RS$$

$$= \log P + \log Q - \log R - \log S; \text{ and so on:}$$

that is, the logarithm of a fraction is equal to the logarithm of its numerator diminished by the logarithm of its denominator: and the operation of *Division* is hereby reduced to that of *Subtraction.*

Hence, the logarithm of a proper fraction is negative, *whilst that of an improper* fraction is positive.

(3) $$\text{Log } P^{\frac{m}{n}} = \frac{m}{n} \log P.$$

As before, $P = a^p$: and $\therefore P^{\frac{m}{n}} = (a^p)^{\frac{m}{n}} = a^{\frac{mp}{n}}$:

whence, $\log P^{\frac{m}{n}} = \frac{m}{n} p = \frac{m}{n} \log P$, where m and n may be any numbers whatever: and thus the operations of *Involution and Evolution* are made to depend upon those of *Multiplication* and *Division* respectively.

94. These fundamental properties of logarithms having been established without reference to any particular value of a, will necessarily be true when the base is 10, as above explained: and we will now exemplify some of their uses, first exhibiting the *forms* in which the numbers and their logarithms are arranged in the *Table of Logarithms* before referred to.

(1) Form for Numbers from 1 to 1200.

No.	Logarithms.	No.	Logarithms.
1	0000000	1196	0777312
2	3010300	1197	0780942
3	4771213	1198	0784568
4	6020600	1199	0788192
5	6989700	1200	0791812

(2) Form for Numbers from 1000 to 108000.

Num.	0	1	2	3	4	5
5400	7323938	4018	4098	4179	4259	4340
1	4742	4822	4903	4983	5063	5144
2	5546	5626	5707	5787	5867	5948
3	6350	6430	6510	6591	6671	6752
4	7153	7234	7314	7394	7475	7555

In the first of these forms, all the mantissæ are given at full length, and it is then only necessary to supply the characteristic: thus, for the logarithm of 1197,

the mantissa is .0780942:

and the characteristic is 3, by Article (90):

$$\therefore \log 1197 = 3.0780942.$$

In the second form, the first three figures of the mantissæ being the same for all the numbers comprised in it, they are inserted only at the beginning, and the rest are supplied according to the values of the digits placed in vertical and horizontal rows: thus, for 54034, we have corresponding to 3 in the *vertical* and 4 in the *horizontal* row, the digits 6671: and this, with 732 which is *common*, gives

.7326671 for the mantissa of 54034:

also, its characteristic being 4, because the number itself is between 10000 and 100000, we have

$$\log 54034 = 4.7326671.$$

It may be observed that the figures in the horizontal row are always continued to 9 in the tables, but 6, 7, 8, 9 are here omitted for want of room.

Examples.

(1) Required the product of 1200 and 4.5.

Here, by means of the former table, we have

$$\log 1200 = 3.0791812:$$

$$\text{and } \log 4.5 = \log \frac{45}{10} = \log \frac{9}{2} = \log 9 - \log 2$$

$$= \log 3^2 - \log 2 = 2 \log 3 - \log 2$$

$$= .9542426 - .3010300 = .6532126:$$

$\therefore$ the logarithm of the required product

$$= 3.0791812 + .6532126 = 3.7323938:$$

and from the latter table, it appears that the mantissa 7323938 belongs to the number 54000: but the characteristic being 3, shews that the product is 5400.

(2) To find the quotient of 54000 by 450, we have $\log 54000 = 4.7323938$, $\log 450 = 2.6532126$, as before:

and therefore the logarithm of the quotient

$$= 4.7323938 - 2.6532126$$

$$= 2.0791812 = \log 120, \text{ by the first table}:$$

or, the quotient is 120.

(3) Given $\log 3.1416 = .4971509$, and $\log 9.86965 = .9943018$, to find the square of 314.16.

Here, the logarithm of the *square* of $314.16 = 2 \times$ logarithm of 314.16, by one of the properties of logarithms:

$$\text{but, } \log 314.16 = \log \{3.1416 \times 100\}$$

$$= \log 3.1416 + \log 100 = .4971509 + 2 = 2.4971509:$$

therefore the logarithm of the required square

$$= 2 \times 2.4971509 = 4.9943018 = \text{the logarithm of } 98696.5,$$

because 4 is the characteristic of the logarithms of all numbers between 10000 and 100000: whence, the square itself is 98696.5 *nearly*: and the deviation of this, from the true value, arises entirely from the circumstance of the mantissæ in the tables not being calculated to a sufficient number of decimal places.

(4) Extract the square root of 31415.92: having $\log 3.141592 = .4971499$, and $\log 1.77245 = .2485749$.

$$\text{Here, } \log 31415.92 = \log \{3.141592 \times 10000\}$$

$$= \log 3.141592 + \log 10000 = 4.4971499:$$

whence, the logarithm of the required *square root*

$$= \tfrac{1}{2}\{4.4971499\} = 2.2485749 = \log 177.245:$$

and the root itself is therefore 177.245 &c.

95. *Given the common logarithm of any numerical magnitude, to find the logarithms of the magnitudes arising from its Multiplication and Division by powers of* 10.

Let $x = \log y$, in the common system: then we have, by the properties in Article (93),

$$\log (y10^m) = \log y + \log 10^m = \log y + m \log 10$$

$$= \log y + m:$$

$$\log \left(\frac{y}{10^m}\right) = \log y - \log 10^m = \log y - m \log 10$$

$$= \log y - m:$$

rom which we perceive that the logarithms of magniudes arising from multiplying or dividing any magnitude y 10^m, are the same as that of the magnitude itself, inreased or diminished by m: in other words, by removng the decimal point of a number m places to the *right* r *left*, we *increase* or *diminish* the logarithm by m.

Thus, by the table in Article (94), $\log 2 = .3010300$:

$$\begin{aligned} \text{whence, } \log 20 &= .3010300 + 1: \\ \log 200 &= .3010300 + 2: \\ \log 2000 &= .3010300 + 3: \ \&c. \\ \log .2 &= .3010300 - 1: \\ \log .02 &= .3010300 - 2: \\ \log .002 &= .3010300 - 3: \ \&c. \end{aligned}$$

96. This proposition shews that, if the logarithm of ny number be once ascertained, the logarithms of all ther numbers whether integral or fractional, consisting f the same *significant* digits in the same order, may be mmediately derived from it by Addition or Subtraction; nd it is this circumstance which enables us to diminish he *Magnitude* of the logarithmic tables without affecting heir *Extent*, by supposing the *mantissa* of every number o be *positive*, and the alteration in the logarithm, coresponding to a change in its local value, to be made by he alteration of the *characteristic* only.

In practice, the characteristics are *prefixed* as whole umbers to the mantissæ, in the same manner as in the rdinary decimal notation, the negative sign being placed *ver* them when they are negative, instead of *before* hem, because it is not intended to have any influence pon the corresponding mantissæ: thus,

$$\begin{aligned} \log 20 &= 1.3010300: & \log .2 &= \bar{1}.3010300: \\ \log 200 &= 2.3010300: & \log .02 &= \bar{2}.3010300: \\ \log 2000 &= 3.3010300: & \log .002 &= \bar{3}.3010300: \ \&c. \end{aligned}$$

97. The peculiar notation above adopted must be arefully attended to, in all the arithmetical operations pon logarithms expressed in the same manner.

(1) By the table of Article (94), we have

$\log 30 = 1.4771213$, and $\log .005 = \bar{3}.6989700$;

$\therefore$ log 30 + log .005 = .4771213 + .6989700 + 1 − 3 = $\bar{1}$.1760913, which may be obtained immediately from the logarithms as they stand, by *adding* together the mantissæ and characteristics, as if they were distinct *algebraical* quantities.

(2) Also, log 30 − log .005 = 1.4771213 − $\bar{3}$.6989700

$$= 1.4771213 + 3 - .6989700 = 4 - .2218487$$

$$= 3 + (1 - .2218487) = 3 + .7781513 = 3.7781513,$$

as will be found by *subtracting* on the same supposition.

In the latter instance, the *negative* mantissa − .2218487 has been converted into the *positive* one + .7781513, by *subtracting* and *adding* 1, so as to be conformable to the tables above explained, and *vice versâ:* and it follows that the same method of merely subtracting the first digit to the right hand from 10, and each of the rest from 9 in order, will be applicable in all other cases, as every mantissa is necessarily less than 1.

(3) Again, since log .02 = $\bar{2}$.3010300, to multiply this logarithm by any number as 4, we have the product

$$= -2 \times 4 + .3010300 \times 4$$

$$= -8 + 1.2041200 = -8 + 1 + .2041200$$

$$= -7 + .2041200 = \bar{7}.2041200,$$

which will also be found by *algebraical* multiplication.

(4) Lastly, to divide log .004 = $\bar{3}$.6020600, by 5, we shall have the quotient = $\frac{1}{5}$ of

$$(-5 + 2.6020600) = -1 + .5204120 = \bar{1}.5204120,$$

the negative characteristic being first made an *exact* multiple of 5, by the subtraction of the number 2 from the characteristic, and its addition to the mantissa.

It is scarcely necessary to remark that these *practical* processes (used in Involution and Evolution) may in every case be superseded by rendering the logarithms altogether *negative*, and then making the mantissæ *positive* in the results of the operations when effected, as intimated above.

98. *To find the relation between the local value of any significant figure, and the characteristic of its logarithm, in the common system.*

The nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9, in the common scale of notation being termed *significant* figures in contradistinction to 0, which has a *local*, but not an *absolute* value, what has been said in Article (90) will enable us readily to perceive the relation mentioned; thus, in 5746.254, which is

$$5.10^3 + 7.10^2 + 4.10^1 + 6.10^0 + 2.10^{-1} + 5.10^{-2} + 4.10^{-3},$$

we have the characteristic

$$\begin{aligned}
&\text{of } 5 \,.\, 10^3 \text{ or } 5000 = +3: \\
&\text{of } 7 \,.\, 10^2 \text{ or } 700 = +2: \\
&\text{of } 4 \,.\, 10^1 \text{ or } 40 = +1: \\
&\text{of } 6 \,.\, 10^0 \text{ or } 6 = 0: \\
&\text{of } 2 \,.\, 10^{-1} \text{ or } .2 = -1: \\
&\text{of } 5 \,.\, 10^{-2} \text{ or } .05 = -2: \\
&\text{of } 4 \,.\, 10^{-3} \text{ or } .004 = -3:
\end{aligned}$$

from which it appears that, whilst the digits in the places of *tens, hundreds, thousands,* &c. have the *positive* characteristics, 1, 2, 3, &c., those in the places of *tenths, hundredths, thousandths,* &c. have the *negative* characteristics, -1, -2, -3, &c., the characteristics of all digits in the *units'* place being 0.

By a generalization of what is observed here, it follows that in *whole numbers,* the characteristic is *positive,* and *less* by 1 than the *number* of figures they contain; whilst in *decimal fractions* it is *negative,* and *greater* by 1 than the number of ciphers which precede the first significant figure.

If the first significant figure be in the n^{th} place to the *left* of the decimal point, its characteristic is $n-1$: but if it be in the n^{th} place to the *right* of that point, its characteristic is $-n$.

99. The *Superiority* of *Briggs's* System of Logarithms to any other that could be adopted, with our present scale of Notation, will hence be manifest.

For, if the base were 6, we should have

$$\log (y 10^m) = \log y + m \log 10:$$

$$\log \left(\frac{y}{10^m}\right) = \log y - m \log 10:$$

and thus be obliged to calculate log 10 in the system whose base is 6, before we could obtain the logarithms of such products and quotients; and the logarithms of numbers consisting of the same digits, but of different local values, would no longer be derivable from each other, by altering the characteristic *only*.

Besides, although the logarithm of 216 would in this system be 3, because $216 = 6^3$; it is not *immediately* perceivable that 216 may be expressed by 6^3, whereas there never can be the least difficulty in seeing that $1000 = 10^3$, or that the common logarithm of 1000 is 3.

The same kind of reasoning is applicable to every other instance and scale; and it may be observed, that were any other System of Notation in use, the most commodious system of logarithms would be that which had the radix of the scale of notation for its base.

100. *To find the mantissæ of the logarithms of numbers consisting of more figures than are comprised in the ordinary tables.*

It has been stated that the mantissa of every number, expressed by the same significant digits in the same order, is the same, without reference to their *local values;* and as the best tables now in use comprize only those of numbers not exceeding *five* figures, continued to *seven* places of decimals, we shall place a *point* after the five *tabular* figures, and also after decimals of the *seventh* order in their mantissæ; and then use the following theorem, which will be established in the *seventh* Chapter.

"If the difference of two *Numbers* be small compared to either of them, that difference will be *nearly* proportional to the difference of their *Logarithms*."

In the following instances, intended to exemplify the Principle just laid down, we shall take only the *mantissæ* of the logarithms of the numbers as we find them in the *tables, neglecting* altogether the *characteristics*, which *may at any time* be supplied by Article (97).

Ex. 1. Given log 10435 = .0184925, and log 10436 = .0185341, to find log 10435.6.

$$\begin{aligned} \text{Since, } \log 10436 &= 0185341 \\ \log 10435 &= 0184925 \end{aligned}$$

$$\therefore \text{ the difference for } 1 = 416;$$

$$\text{and the difference for } .1 = \frac{1}{10}(416) = 41.6;$$

whence, the difference for $.6 = 41.6 \times 6 = 249.6 = 250$. nearly, by taking the nearest integer: and log 10435.6 = log 10435 + 250., where the point in the latter indicates that seven places of decimals have preceded it: that is,

$$\begin{aligned} \log 10435.6 &= .0184925 \\ &+ \quad 250 \\ &= .0185175 \end{aligned}$$

so that the required mantissa is .0185175, and therefore the logarithm of the number proposed will be 4.0185175.

Hence, log 104356 = 5.0185175, from merely increasing the *characteristic* by 1: and so on.

Ex. 2. Given that the difference for log 34092 = 5326525, is 127, find log 3409264.

Here, Diff. for .1 = 12.7, and Diff. for .01 = 1.27: whence, proceeding as before, we shall have

Numbers.	*Logarithms.*
34092	5326525
6	76.2
4.........	5.08
∴ log 3409264 =	5326606.28

or, the mantissa of the number 3409264 to *seven* places of decimals is 5326606: and of course the *entire* logarithm will be determined by prefixing the proper *characteristic*:

$$\begin{aligned} \text{thus, } \log\ 3409.264 &= 3.5326606: \\ \log\ 3.409264 &= 0.5326606: \\ \log\ .03409264 &= \bar{2}.5326606. \end{aligned}$$

In the tables there is generally placed a small column of numbers marked *D*, denoting the differences of two *consecutive* logarithms at certain intervals, and the figures to be added to the tabular mantissæ are had at once from what is called a table of *proportional parts* affixed to the right of it, in which the multiplications above used have been effected: and the only thing then requiring attention will be their *positions*, which must depend upon those of the digits to which they respectively belong as is seen in the preceding examples.

The last result might have been obtained from a *proportion:* for,

$$1 : .64 :: 127 : 81.28;$$

and $81.28 = 76.2 + 5.08$, as above found: but the *practical* convenience of the former method is readily perceived, and it is manifestly applicable, *whatever* be the number of figures concerned.

101. *To find the number corresponding to a logarithm not found exactly in the tables.*

This is the reverse of the preceding proposition, and will be effected by means of the same principles, as we shall shew in the following instances.

Ex. 1. Given $\log 1.0686 = .0288152$, and $\log 1.0687 = .0288558$, to find the number whose logarithm is .0288355.

Here, Diff. for $1 = 406$: and the Difference between the less and the proposed logarithm $= 203$:

whence, $406 : 203 :: 1 : .5$, by the last Article:

that is, 5 is the *additional* digit, corresponding to the logarithm given, so that

$$\log 1.06865 = .0288355.$$

Whatever positions the points in the proposed numbers may occupy, the number required will consist of the same figures, and its absolute value will have to be decided by the characteristic of its logarithm.

Ex. 2. If $\log 17122 = 2335545$, and the difference $= 254$: find the number whose logarithm is 1.2335678.

First, to make a table of proportional parts, we consider *that* by removing the point in 254, one, *two*, &c.

places to the left, we obtain the differences for .1, .01, &c: and then we shall have

$$25.4 \times 2 = 50.8 = 51 : \; 25.4 \times 3 = 76.2 = 76 :$$
$$25.4 \times 4 = 101.6 = 102 : \; 25.4 \times 5 = 127.0 = 127 :$$
$$25.4 \times 6 = 152.4 = 152 : \; 25.4 \times 7 = 177.8 = 178 :$$
$$25.4 \times 8 = 203.2 = 203 : \; 25.4 \times 9 = 228.6 = 229 :$$

where the *nearest* integers are always adopted.

Now the difference between the two logarithms, to be attended to here, is 133: and

for .5, the difference is 127

60

for .02, the difference is 51

90

for .003, the difference is 76

14, &c.

and ∴ the number whose logarithm is 1.2335678 will be 17.122523 &c., in which the characteristic 1, directs the position of the decimal point.

These two articles must necessarily claim the attention of the *practical* mathematician; and the reasons of them will be fully investigated and considered in Chapter VII.

III. *The Application of Logarithms to Trigonometrical Formulæ: and the Adaptation of Formulæ to Logarithmic Computation by the Use of Subsidiary Angles.*

102. The Construction and Verification of the Trigonometrical Canon have been already explained, and it has been seen in Article (93), that Logarithms afford great facilities in the *Multiplication*, *Division*, *Involution*, and *Evolution* of numbers consisting of many places of figures, though not in their *Addition* and *Subtraction*: hence, the *Logarithms of the Angular Functions* for every *Minute* are also tabulated, forming *a Table of Logarithmic Sines, Cosines*, &c.

Now, since many of these functions are less than 1, and for *certain values of the angle*, would be expressed by decimals commencing with several places of ciphers, the natural sines, &c. in the *Tables* are sometimes made 10000 times as great as they ought to be: and therefore whenever they are used, the decimal points must at once be removed *four* places to the left hand to obtain their true values. In all such instances, the corresponding logarithms would be *negative* quantities: and accordingly each of the sines, cosines, &c. is increased *ten thousand million* times; or, which is the same thing, is calculated *to the radius* 10^{10}, in order to avoid negative quantities, which would destroy the simplicity of the Tables: and hence it follows that each of the *Tabular Logarithmic Sines, Cosines*, &c. is greater by 10 than its *true* value, because $\log 10^{10} = 10$.

It will consequently be necessary, in the application of these tables to numerical examples, to subtract 10 from each of the logarithmic sines, cosines, &c. concerned in them: and this may evidently be done by subtracting from either member of an equality, a number of *tens* equal to the excess of its dimensions above that of the other, as will appear from rendering the members homogeneous by means of powers of the radius, or of 10^{10}.

Ex. 1. We have seen that $\tan A = \dfrac{\sin A}{\cos A}$: whence, taking the *tabular* logarithms of both members, and denoting them by $L \sin A$, $L \cos A$, &c., we have

$$\begin{aligned} L \tan A - 10 &= \{L \sin A - 10\} - \{L \cos A - 10\} \\ &= L \sin A - 10 - L \cos A + 10 \\ &= L \sin A - L \cos A: \end{aligned}$$

$$\therefore L \tan A = 10 + L \sin A - L \cos A:$$

which is the result that may be obtained from the equality,

$$\frac{\tan A}{10^{10}} = \frac{\sin A}{\cos A}, \text{ or } \tan A = 10^{10} \frac{\sin A}{\cos A},$$

by considering the *tabular* logarithms as the *true* logarithms of *all* the quantities employed.

Ex. 2. If $x = a \sin^2 A$: we shall have

$$\log x = \log a + 2\{L \sin A - 10\} = \log a + 2 L \sin A - 20,$$

which is the true numerical value of $\log x$: in fact, to the *tabular radius* 10^{10}, the equation is

$$x = a\frac{\sin^2 A}{10^{20}}\text{, or } x10^{20} = A\sin^2 A:$$

and it gives the same value for log x, if the tabular logarithm of the sine be now considered its true logarithm.

Ex. 3. From the equation $y^{\frac{1}{2}} = \sin A\,(\cos B)^{\frac{3}{4}}(\tan C)^{-\frac{4}{3}}$, we find $\frac{1}{2}\log y$

$$= (L\sin A - 10) + \frac{3}{4}(L\cos B - 10) - \frac{4}{3}(L\tan C - 10),$$

which is easily simplified: and this *direct* process will generally be attended with less labour, than if 10^{10} or the tabular radius were introduced, as in the preceding instances: but the same result will be immediately derived from the *rationalized* equation,

$$y^6 = (\sin A)^{12}(\cos B)^9(\tan C)^{-16} = \frac{(\sin A)^{12}(\cos B)^9}{(\tan C)^{16}}.$$

103. When an expression is made up of *factors* either *direct* or *inverse:* that is, when it is in the form of a *product* or a *quotient* of *monomial* quantities, the properties of logarithms are immediately applicable to it, and it is said to be *adapted to Logarithmic Computation*, as in the examples above used.

If however the expression be formed of different parts connected by the sign + or −, it will be necessary to make it assume one of the forms above mentioned, before the properties of logarithms can be of any avail: and this is done either by means of the *Transformations* before investigated, or by the use of *Subsidiary Angles*, which are angles introduced merely to effect this purpose, and have no reference to any other consideration in the quantities proposed; and both of these will be seen in the following examples.

Ex. 1. If $x = \sin A + \sin B$, we have, by Article (55),

$$x = 2\sin\tfrac{1}{2}(A + B)\cos\tfrac{1}{2}(A - B):$$

$$\therefore \log x = \log 2 + \{L\sin\tfrac{1}{2}(A + B) - 10\}$$
$$+ \{L\cos\tfrac{1}{2}(A - B) - 10\}$$
$$= \log 2 + L\sin\tfrac{1}{2}(A + B) + L\cos\tfrac{1}{2}(A - \mathrm{B}) - 20:$$

from which log x, and thence x itself may be found.

Ex. 2. If $x^2 = \dfrac{\sin\alpha - \sin\beta}{\sin(\alpha+\beta)}$, we shall have

$$x^2 = \frac{2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}{2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha+\beta)} = \frac{\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)}:$$

$$\therefore\ 2\log x = \left\{L\sin\tfrac{1}{2}(\alpha-\beta) - 10\right\} - \left\{L\sin\tfrac{1}{2}(\alpha+\beta) - 10\right\}$$
$$= L\sin\tfrac{1}{2}(\alpha-\beta) - L\sin\tfrac{1}{2}(\alpha+\beta):$$

from which x may be found by the tables.

Ex. 3. Let $x = a \pm b$, be adapted to logarithms.

(1) $$x = a + b = a\left\{1 + \frac{b}{a}\right\}.$$

Here, assuming the *subsidiary angle* θ, such that $\tan^2\theta = \dfrac{b}{a}$, in order to render $\left(1 + \dfrac{b}{a}\right)$ a *monomial*,

$$\text{we shall have } x = a\{1 + \tan^2\theta\} = a\sec^2\theta:$$
$$\text{and } \therefore\ \log x = \log a + 2L\sec\theta - 20:$$

where θ will be found from the equation $\tan^2\theta = \dfrac{b}{a}$:

$$\text{or, } 2L\tan\theta = 20 + \log b - \log a.$$

(2) $$x = a - b = a\left\{1 - \frac{b}{a}\right\}.$$

Here, if θ be such an angle that $\cos^2\theta = \dfrac{b}{a}$, we have

$x = a\{1 - \cos^2\theta\} = a\sin^2\theta$, a monomial quantity; and the two equations for determining x will be

$$L\cos\theta = \tfrac{1}{2}\{\log b - \log a + 20\}:$$
$$\log x = \log a + 2L\sin\theta - 20.$$

Ex. 4. Adapt $x = \sqrt{a^2 \pm b^2}$, to logarithms.

(1) $$x = \sqrt{a^2 + b^2} = a\sqrt{1 + \left(\frac{b}{a}\right)^2}.$$

Whatever be the relative values of a and b, we may assume $\tan\theta = \dfrac{b}{a}$, because the *Tangent* admits of all *degrees of* magnitude from 0 to ∞:

$$\therefore\ x = a\sqrt{1+\tan^2\theta} = a\sec\theta :$$

nd x will be found by means of the two equations:

$$L\tan\theta = \log b - \log a + 10 :$$
$$\log x = \log a + L\sec\theta - 10.$$

(2) $$x = \sqrt{a^2 - b^2} = a\sqrt{1 - \left(\frac{b}{a}\right)^2}.$$

In this case, when x is a real magnitude, $\frac{b}{a}$ must necessarily be less than 1, and may therefore be represented by $\cos\theta$, because the *Cosine* may be of any magnitude between 0 and 1:

$$\therefore\ x = a\sqrt{1-\cos^2\theta} = a\sin\theta :$$

nd the two equations for finding x will be

$$L\cos\theta = \log b - \log a + 10 :$$
$$\log x = \log a + L\sin\theta - 10.$$

Ex. 5. If $x = a \pm \sqrt{a^2 - b^2} = a\left(1 \pm \sqrt{1 - \frac{b^2}{a^2}}\right)$: by ssuming $\sin\theta = \frac{b}{a}$, we shall have

$$x = a\,(1 \pm \cos\theta) = 2a\cos^2\tfrac{1}{2}\theta,\ \text{and}\ 2a\sin^2\tfrac{1}{2}\theta ;$$

nd the logarithmic computations are readily effected.

Ex. 6. If $x = a \pm \sqrt{a^2 + b^2} = a\left(1 \pm \sqrt{1 + \frac{b^2}{a^2}}\right)$: by putting $\tan\theta = \frac{b}{a}$, we find $x = a\,\{1 \pm \sec\theta\}$, which are asily made to assume the forms

$$x = a\tan\theta\cot\tfrac{1}{2}\theta,\ \text{and}\ -x = a\tan\theta\tan\tfrac{1}{2}\theta :$$

nd they are both adapted to logarithms.

Ex. 7. From $x = \sqrt{a+b} \pm \sqrt{a-b}$, we shall have

$$x = \sqrt{a}\left\{\sqrt{1 + \frac{b}{a}} \pm \sqrt{1 - \frac{b}{a}}\right\}$$
$$= \sqrt{a}\,\{\sqrt{1+\sin\theta} \pm \sqrt{1-\sin\theta}\},\ \text{if}\ \sin\theta = \frac{b}{a},$$

$2\sqrt{a}\cos\frac{1}{2}\theta$, *and* $2\sqrt{a}\sin\frac{1}{2}\theta$, by Article (48).

Ex. 8. Let $x = \sqrt{\dfrac{a+b}{a-b}} \pm \sqrt{\dfrac{a-b}{a+b}} = \dfrac{2a}{\sqrt{a^2-b^2}}$, and $\dfrac{2b}{\sqrt{a^2-b^2}}$: then, making $\dfrac{b}{a} = \sin\theta$, we shall have

$$x = \frac{2}{\sqrt{1-\dfrac{b^2}{a^2}}} = \frac{2}{\cos\theta} = 2\sec\theta, \text{ in the first case:}$$

also, putting $\dfrac{a}{b} = \sec\phi$, we obtain

$$x = \frac{2}{\sqrt{\sec^2\phi - 1}} = \frac{2}{\tan\phi} = 2\cot\phi, \text{ in the second case.}$$

In all such instances as those above given, the *adequacy* of the function of the subsidiary angle assumed, to represent the quantity under consideration, must be carefully attended to.

Ex. 9. To adapt $a\sin x \pm b\cos x = c$, to logarithmic computation, for the determination of x.

Here, $\sin x \pm \dfrac{b}{a}\cos x = \dfrac{c}{a}$, by dividing by a:

$$\therefore \text{ if } \tan\theta = \frac{b}{a}, \text{ we have } \sin x \pm \tan\theta\cos x = \frac{c}{a}:$$

$$\text{or, } \sin x\cos\theta \pm \cos x\sin\theta = \frac{c}{a}\cos\theta:$$

$$\text{or, } \sin(x \pm \theta) = \frac{c}{a}\cos\theta:$$

and the equations for finding x will therefore be

$$L\tan\theta = \log b - \log a + 10:$$

$$L\sin(x \pm \theta) = \log c - \log a + L\cos\theta:$$

for, θ being given by the former equation, $x \pm \theta$ and therefore x will be determined from the latter.

$$\text{Dividing by } b,\ \frac{a}{b}\sin x \pm \cos x = \frac{c}{b}:$$

$$\therefore \text{ if } \cot\theta = \frac{a}{b}, \text{ we shall have}$$

$$\sin x \cos \theta \pm \cos x \sin \theta = \frac{c}{b} \sin \theta: \text{ or, } \sin (x \pm \theta) = \frac{c}{b} \sin \theta:$$

$$\text{and } L \cot \theta = \log a - \log b + 10:$$

$$L \sin (x \pm \theta) = \log c - \log b + L \sin \theta:$$

are the equations for finding x: which manifestly amount to the same thing as those before given, since

$$\tan \theta \cot \theta = 1, \text{ and } \therefore L \cot \theta = 20 - L \tan \theta; \text{ \&c.}$$

These adaptations are of very extensive utility, as numerous instances occur in practice, where it is required to determine an angle by means of an expression which involves its sine and cosine connected by addition or subtraction.

Other instances will be seen in the *Solution of Triangles;* and it may be observed that a judicious selection of a subsidiary angle, which generally depends upon a dexterity in the use of trigonometrical formulæ, will contribute greatly to facilitate the computation.

104. Since the logarithmic functions of an angle are registered for every *minute* only in the common tables, those of an angle containing *seconds*, &c. are determined from the Principle enunciated in Article (100), as will be explained in the following examples: and *vice versâ.*

Ex. 1. If $L \sin 53^\circ 7' = 9.9030136$, and $L \sin 53^\circ 8' = 9.9031084$, find $L \sin 53^\circ 7' 48''.36$.

Here, Diff. for $1'$ or $60'' = 948$, $\therefore$ Diff. for $10'' = 158$: whence, attending to the suggestions of Article (100), we have

$$\begin{array}{ll} L \sin 53^\circ 7' & = 9.9030136 \\ \text{Diff. for } 40'' = & \quad\;\; 632 \\ \text{Diff. for } 8'' = & \quad\;\; 126.4 \\ \text{Diff. for } .3'' = & \quad\;\;\;\; 4.74 \\ \text{Diff. for } .06'' = & \quad\;\;\;\;\; .948 \\ \hline \end{array}$$

$$\therefore L \sin 53^\circ 7' 48''.36 = 9.9030900.088;$$

that is, $L \sin 53^\circ 7' 48''.36 = 9.9030900$,

to the usual extent of *seven* places of decimals.

In the same way, from $L\tan 53^\circ\,7' = 10.1247266$, and $L\tan 53^\circ\,8' = 10.1249898$, we shall find $L\tan 53^\circ\,7'\,48''.36 = 10.1249837$.

Ex. 2. Given $L\cos 36^\circ\,52' = 9.9031084$, and $L\cos 36^\circ\,53' = 9.9030136$, to find $L\cos 36^\circ\,52'\,11''.64$.

Here, Diff. for $60'' = -948$, $\therefore$ Diff. for $10'' = -158$: from which are immediately obtained

$$\begin{array}{lr}
L\cos 36^\circ\,52' = & 9.9031084 \\
\text{Diff. for } 10'' = & -158 \\
\text{Diff. for } 1'' = & -15.8 \\
\text{Diff. for } .6'' = & -9.48 \\
\text{Diff. for } .04'' = & -.632 \\
\hline
\end{array}$$

whence, $L\cos 36^\circ\,52'\,11''.64 = 9.9030900.088$.

In this example it will be seen at once that the angles concerned are the *Complements* of those in the last, and the *same* result might therefore have been anticipated: but the *practical* conclusion is, that whenever the *complement* of an angle occurs in the *place* of the angle *itself*, the *subtraction* of the proportional parts must be implied, instead of their *addition*.

This may be verified by slightly altering the *form* in which the numbers are placed: for taking

$36^\circ\,52'\,11''.64 = 36^\circ\,53' - 48''.36$, we have

$$\begin{array}{lr}
L\cos 36^\circ\,53' = & 9.9030136 \\
\text{Diff. for } -40'' = & 632 \\
\text{Diff. for } -8'' = & 126.4 \\
\text{Diff. for } -.3'' = & 4.74 \\
\text{Diff. for } -.06'' = & .948 \\
\hline
\end{array}$$

$\therefore L\cos 36^\circ\,52'\,11''.64 = 9.9030900.088$

and the process is perhaps rather simpler than the preceding one, as the logarithm requires only the operation of *Addition:* also, it dispenses with the necessity of using the *Arithmetic Complements* for negative proportional parts which are obtained by diminishing the characteristic *by 1, and* then subtracting the last decimal figure from [illegible] *and all the* rest from 9 in succession.

Ex. 3. Given $L\tan 6^\circ 30' = 9.0566595$, and $L\tan 6^\circ 31' = 9.0577813$, to find the angle whose $L\tan = 9.0571285$.

Here, Diff. for $60'' = 11218$, $\therefore$ Diff. for $10'' = 1869.6$:

also, the given difference $= 4690$:

whence, $1869.6 : 4690 :: 10'' : 25''$, nearly:

and therefore we have $9.0571285 = L\tan 6^\circ 30' 25''$.

With a table of proportional parts at hand, we might proceed in the following form:

$$\begin{array}{rrl} & 1869.6 & = \text{Diff. for } 10'': \\ \text{also,} & 4690 & = \text{Diff. given} \\ & 3739.2 & = \text{Diff. for } 20'' \\ \hline & 950.8 & \\ & 934.8 & = \text{Diff. for } 5'' \\ \hline & 16 & \end{array}$$

and in both these operations, the approximation may be made to any degree of nicety we please.

Ex. 4. If $L\cot 83^\circ 29' = 9.0577813$, and $L\cot 83^\circ 30' = 9.0566595$, required the value of A,

when $L\cot A = 9.0571269$.

Here, Diff. for $60'' = -11218$;

$\therefore$ Diff. for $-10'' = 1869.6$,

and the given Diff. $= 4674$, in this case;

$$\begin{array}{rr} \therefore\ L\cot 83^\circ 30' = & 9.0566595 \\ \text{Diff. for } -20'' = & 3739.2 \\ \text{Diff. for } -5'' = & 934.8 \\ \hline \therefore\ L\cot 83^\circ 29' 35'' = & 9.0571269 \end{array}$$

that is, $9.0571269 = L\cot 83^\circ 29' 35''$: and applying the observations in the second example, we are apprized of a slight discrepancy in this result from what we might have been led to expect by Ex. 3: but in it the *decimals* of a second have been disregarded, which may generally be done when great nicety is not required.

105. *However, the* Principle of Proportional Parts *cannot always be* depended upon, even for seconds, when

the functions attain nearly their extreme values, as will be proved hereafter: but the tables in use will present this circumstance to the student's notice at once: thus,

$$L\sin 88^\circ 48' = 9.9999047:$$
$$L\sin 88^\circ 49' = 9.9999074:$$
$$L\sin 88^\circ 50' = 9.9999100:$$
$$L\sin 88^\circ 51' = 9.9999125:$$

shew that the uniform difference of 1′ in the angles gives 27, 26, and 25, for the differences in their logarithmic functions; and we shall merely observe here, that when an angle A is required to be found from its logarithmic functions, it may be had with tolerable accuracy, by means of this principle, in all cases where they do not approach near to their greatest or least values.

THE SOLUTION OF TRIANGLES.

106. Using the notation adopted in the last chapter, we have in every triangle;

$$A + B + C = 180^\circ:$$

$$\frac{\sin A}{\sin B} = \frac{a}{b}, \quad \frac{\sin A}{\sin C} = \frac{a}{c}, \quad \frac{\sin B}{\sin C} = \frac{b}{c}:$$

where any one of these latter equalities is immediately derivable from the two others: and therefore we have *three* independent equations, by means of which to determine three of the quantities concerned, when the remaining three are given.

Since, however, the *first* equation above cited is independent of the *sides* of the triangle, it follows that the remaining *two* will not be sufficient for determining the *magnitudes* of the *three* sides, though they adequately express their *ratios;* that is, when the *three angles* are given, there is no solution of the triangle; and this is also evident from the circumstance that, whilst the lengths of the sides of a triangle are all increased or diminished *in the same* proportion, the magnitudes of the angles *remain the same*.

In all other cases, *three* data will be sufficient for a mplete solution of the triangle, with one partial ex-ption, which will hereafter be fully explained.

I. *Right-angled Triangles.*

107. Here the *right angle* being given, any *two* of e parts, except the *two angles*, will, by means of the uations above mentioned, give the magnitudes of the st, as will be made to appear in the following *Cases*.

(1) *Given the side* a, *and the opposite angle* A: *to d the rest.*

If C be the right angle, we have $A + B = 90^\circ$:

$$\text{whence, } B = 90^\circ - A;$$

$$= \frac{\sin B}{\sin A} = \frac{\cos A}{\sin A} = \cot A = \frac{1}{\tan A}; \ \therefore\ b = a \cot A = \frac{a}{\tan A}:$$

$$\frac{c}{a} = \frac{\sin C}{\sin A} = \frac{1}{\sin A} = \operatorname{cosec} A; \ \therefore\ c = \frac{a}{\sin A} = a \operatorname{cosec} A.$$

In Logarithms:

$$\log b = \log a + L \cot A - 10 = \log a - L \tan A + 10:$$

$$\log c = \log a - L \sin A + 10 = \log a + L \operatorname{cosec} A - 10.$$

(2) *Given the side* b, *and the adjacent angle* A: *to d the rest.*

Here, $B = 90^\circ - A$:

$$\frac{a}{b} = \frac{\sin A}{\sin B} = \frac{\sin A}{\cos A} = \tan A, \text{ or } a = b \tan A:$$

$$\frac{c}{b} = \frac{\sin C}{\sin B} = \frac{1}{\cos A} = \sec A, \text{ or } c = b \sec A.$$

In Logarithms:

$$\log a = \log b + L \tan A - 10:$$

$$\log c = \log b + L \sec A - 10.$$

(3) *Given the side* c, *and the adjacent angle* A: *to d the rest.*

Here, $B = 90^\circ - A$:

$$\frac{a}{c} = \frac{\sin A}{\sin C} = \sin A, \text{ or } a = c \sin A:$$

(5)

Logarithms.

$L \sin A$ – log [illegible]

log a – $L \cot A - 10 =$ log [illegible] cos A –

he preceding case, the value of [illegible]

$b^2 - c^2 - a^2$, which has been ada

garithmic computation in Ex. 4, of page (78): and when is nearly equal to a, or b is very small, this will be the tter practical solution, for the reason mentioned in rticle (105).

108. COR. It will be shewn in the seventh chapter, at if h denote a small given increment of an angle, and $\sin A$, $\triangle \cos A$, $\triangle \tan A$, $\triangle \cot A$, represent the corres-onding increments of the functions of A, the following ill be good approximations:

$\sin A = \cos A \sin h$, when A is not nearly $= 90^\circ$:

$\cos A = - \sin A \sin h$, when A is not nearly $= 0^\circ$ or 180°:

$\tan A = \sec^2 A \tan h$, when A is not nearly $= 90^\circ$:

$\cot A = - \operatorname{cosec}^2 A \tan h$, when A is not nearly $= 0^\circ$ or 180°.

Whence, if A be very small, $\cos A$ and $\sec A$ will be ry nearly $= 1$, an invariable quantity, and we shall ve

$$\triangle \sin A \propto \sin h, \text{ and } \triangle \tan A \propto \tan h, \text{ nearly};$$

that the principle of proportional parts may here be pended upon both for the natural and logarithmic *sine* d *tangent*.

But if A be nearly a right angle, we have

$$\triangle \cos A \propto - \sin h, \text{ and } \triangle \cot A \propto - \tan h, \text{ nearly},$$

r a similar reason: and the same will therefore now ld for the *cosine* and *cotangent*.

To apply these considerations to the solution of the st case, we should consequently use, when a is much less an c, the expression $\sin A = \frac{a}{c}$, above given:

$$\text{and } \therefore L \sin A = \log a - \log c + 10:$$

t when a is nearly equal to c, the solution may be odified, as follows;

$$\tan \tfrac{1}{2} B = \sqrt{\frac{1 - \cos B}{1 + \cos B}} = \sqrt{\frac{1 - \sin A}{1 + \sin A}} = \sqrt{\frac{c - a}{c + a}};$$

$$\therefore L \tan \tfrac{1}{2} B = 10 + \tfrac{1}{2} \{\log (c - a) - \log (c + a)\}$$

$$= 10 - \tfrac{1}{2} \{\log (c + a) - \log (c - a)\} = L \tan (45^\circ - \tfrac{1}{2} A).$$

II. *Oblique-Angled Triangles.*

109. Here we must have three parts given, which are not all angles, to find the rest; and the solutions will be comprehended in the following cases.

(1) *Given the side* a, *and the angles* A, B: *to find the rest.*

Since, $A+B+C=180°$, we have $C=180°-(A+B)$:

$$\frac{b}{a}=\frac{\sin B}{\sin A},\ \text{ or }\ b=a\frac{\sin B}{\sin A}:$$

$$\frac{c}{a}=\frac{\sin C}{\sin A},\ \text{ or }\ c=a\frac{\sin C}{\sin A}.$$

In Logarithms:

$$\log b=\log a+L\sin B-L\sin A:$$

$$\log c=\log a+L\sin C-L\sin A.$$

(2) *Given the side* c, *and the adjacent angles* A, B: *to find the rest.*

Here, $C=180°-(A+B)$, which is therefore known:

$$\frac{a}{c}=\frac{\sin A}{\sin C},\ \text{ or }\ a=c\frac{\sin A}{\sin C}:$$

$$\frac{b}{c}=\frac{\sin B}{\sin C},\ \text{ or }\ b=c\frac{\sin B}{\sin C}.$$

In Logarithms:

$$\log a=\log c+L\sin A-L\sin C:$$

$$\log b=\log c+L\sin B-L\sin C.$$

(3) *Given the sides* a, b, *and the angle* A *opposite one of them: to find the rest.*

Here, $\frac{\sin B}{\sin A}=\frac{b}{a}$, or $\sin B=\frac{b}{a}\sin A$:

and C is then found from $C=180°-(A+B)$:

$$\frac{c}{a}=\frac{\sin C}{\sin A},\ \text{ or }\ c=a\frac{\sin C}{\sin A}.$$

In Logarithms:

$$L\sin B=\log b-\log a+L\sin A:$$

$$\log c=\log a+L\sin C-L\sin A.$$

110. This case is the partial exception before alluded o, the solution depending entirely upon the determination f the angle B, which may be doubtful, unless the given ngle be *obtuse:* for, the sine of an angle being equal to he sine of its supplement, the equation $\sin B = \frac{b}{a} \sin A$, annot, without other considerations, inform us whether B is greater or less than 90°.

(1) Of the given sides, let a be greater than b: then A is greater than B, or B is less than A: also, $A + B$ is ess than 180°:

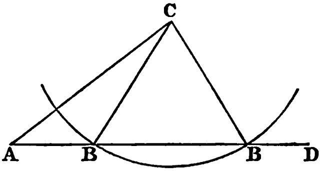
$\therefore$ since B is less than A, $2B$ is less than $A + B$ and herefore *a fortiori* less than 180°:

whence B is less than 90°, or is acute:

nd the solution is complete and determinate.

(2) Let a be less than b, or b greater than a: then B s greater than A, and $2B$ is greater than $A+B$: and ince $A+B$ is less than 180°, $2B$ may be greater than $A+B$, whether it be less or greater than 180°, or whether B be less or greater than 90°: that is, B may be either acute or obtuse, and there are therefore *two* solutions of the triangle.

This latter is called the *Ambiguous Case*, and its circumstances will easily be understood by means of the ollowing construction.



At the point A in the indefinite straight line AD, make the $\angle DAC$ equal to the given $\angle A$: take $AC = b$, and with centre C and radius $= a$, describe a circular arc, which will evidently cut AD in two points B, B on the *same* side of A, since a is less than b, or CB less than CA: *that is, each of the triangles* ABC *possesses the same data,* and consequently each of the required parts admits of two different values, the angles at B being *supplemental* to each other.

The construction also shews that, if CB be greater than CA, there can be only one solution, because the intersections B, B will then be on *opposite* sides of A, and therefore the *given* angle A is not one of the data of the triangle. If the circle *touch* the line AD, the triangle is *right-angled* at B, and there is no ambiguity; but if it do not *meet* AD, no triangle can be formed, and the data are *incongruous*.

(3) *Given the sides* a, b, *and the included angle* C: *to find the rest.*

By Article (70), we have $\dfrac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)} = \dfrac{a-b}{a+b}$:

$$\text{or,}\ \tan \tfrac{1}{2}(A-B) = \frac{a-b}{a+b} \tan \tfrac{1}{2}(A+B) = \frac{a-b}{a+b} \cot \tfrac{1}{2}C,$$

$$\text{since } \tfrac{1}{2}(A+B) = 90 - \tfrac{1}{2}C:$$

$\therefore\ \frac{1}{2}(A-B)$ is determined: and $\frac{1}{2}(A+B)$ is given:

$$\left.\begin{aligned}\text{whence, } A &= \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B)\\ \text{and } B &= \tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B)\end{aligned}\right\}\ \text{are found}:$$

$$\text{also, } \frac{c}{a} = \frac{\sin C}{\sin A},\ \text{ or } c = a\,\frac{\sin C}{\sin A}.$$

In Logarithms:

$$L \tan \tfrac{1}{2}(A-B) = \log(a-b) - \log(a+b) + L \cot \tfrac{1}{2}C:$$

$$\log c = \log a + L \sin C - L \sin A.$$

If a and b involve angular functions, or be given only by their logarithms, we may adapt the former equation to logarithms by assuming $\tan \theta = \dfrac{a}{b}$:

$$\text{for,}\quad \tan \tfrac{1}{2}(A-B) = \frac{\dfrac{a}{b} - 1}{\dfrac{a}{b} + 1} \cot \tfrac{1}{2}C = \frac{\tan \theta - 1}{\tan \theta + 1} \cot \tfrac{1}{2}C$$

$$= \tan(\theta - 45^\circ) \cot \tfrac{1}{2}C:\ \text{and we have}$$

$$L \tan \theta = \log a - \log b + 10:$$

$$L \tan \tfrac{1}{2}(A-B) = L \tan(\theta - 45^\circ) + L \cot \tfrac{1}{2}C - 10.$$

If a be less than b, then A is less than B, and

$$\tan \tfrac{1}{2}(B-A) = \frac{b-a}{b+a} \cot \tfrac{1}{2} C$$

$$= \frac{1-\tan\theta}{1+\tan\theta} \cot \tfrac{1}{2} C = \tan(45^\circ - \theta) \cot \tfrac{1}{2} C, \text{ as before.}$$

Also, since $\dfrac{c}{a+b} = \dfrac{\sin C}{\sin A + \sin B} = \dfrac{\sin(A+B)}{\sin A + \sin B}$

$$= \frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A+B)}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)} = \frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}$$

$$= \frac{\sin\frac{1}{2}C}{\cos\frac{1}{2}(A-B)}, \text{ we have } c = \frac{(a+b)\sin\frac{1}{2}C}{\cos\frac{1}{2}(A-B)}:$$

similarly, it is found that $c = \dfrac{(a-b)\cos\frac{1}{2}C}{\sin\frac{1}{2}(A-B)}$:

nd the value of c may be obtained from the one best
ited to the circumstances of the case.

111. The value of the side c may be found at once
om the equation $c^2 = a^2 + b^2 - 2ab\cos C$, which is easily
dapted to logarithmic computation, in each of the fol-
wing forms.

(1) $$c^2 = a^2 + b^2 - 2ab\cos C$$

$$= a^2 + b^2 - 2ab(1 - 2\sin^2 \tfrac{1}{2} C)$$

$$= a^2 - 2ab + b^2 + 4ab\sin^2 \tfrac{1}{2} C = (a-b)^2 + 4ab\sin^2 \tfrac{1}{2} C$$

$$= (a-b)^2 \left\{1 + \frac{4ab}{(a-b)^2} \sin^2 \tfrac{1}{2} C\right\};\ \therefore\ \frac{4ab}{(a-b)^2} \sin^2 \tfrac{1}{2} C,$$

hich is necessarily *positive*, and may be of *any* magni-
ude, being assumed $= \tan^2\theta$, we shall have

$$c^2 = (a-b)^2(1+\tan^2\theta) = (a-b)^2\sec^2\theta:$$

$$\text{or, } c = (a-b)\sec\theta:$$

also, from $\tan\theta = \dfrac{2\sqrt{ab}}{a-b} \sin \tfrac{1}{2} C$, we obtain

$L \tan\theta = \log 2 + \tfrac{1}{2}\log a + \tfrac{1}{2}\log b - \log(a-b) + L\sin\tfrac{1}{2}C$,
hich gives the value of θ:

log $c =$ *log* $(a-b) + L \sec\theta - 10$, gives the value of c.

The eligibility of this solution depends upon the value of $\tan\theta$ not being very large: which will always be the case when $a-b$ is not small when compared with either a or b. See Articles (105) and (108).

(2)
$$\begin{aligned} c^2 &= a^2+b^2-2ab\cos C \\ &= a^2+b^2-2ab\,(2\cos^2\tfrac{1}{2}C-1) \\ &= a^2+2ab+b^2-4ab\cos^2\tfrac{1}{2}C \\ &= (a+b)^2\left\{1-\frac{4ab}{(a+b)^2}\cos^2\tfrac{1}{2}C\right\}: \end{aligned}$$

and because $\sqrt{ab}$ can never be greater than $\frac{1}{2}(a+b)$,

$$\frac{4ab}{(a+b)^2}\cos^2\tfrac{1}{2}C \text{ can never be greater than } 1:$$

$$\therefore\ \frac{4ab}{(a+b)^2}\cos^2\tfrac{1}{2}C \text{ may be assumed} = \sin^2\theta:$$

$$\text{whence, } c^2=(a+b)^2(1-\sin^2\theta)=(a+b)^2\cos^2\theta:$$

$$\text{or, } c=(a+b)\cos\theta:$$

and the logarithmic computation is similar to the last.

This solution evades the inconvenience arising from the trigonometrical tables, when a and b are nearly equal to each other: and it may always be used unless $\sin^2\theta$ nearly $=1$. See Articles (105) and (108).

(3)
$$\begin{aligned} c^2 &= (a^2+b^2)(\cos^2\tfrac{1}{2}C+\sin^2\tfrac{1}{2}C)-2ab(\cos^2\tfrac{1}{2}C-\sin^2\tfrac{1}{2}C) \\ &= (a^2+2ab+b^2)\sin^2\tfrac{1}{2}C+(a^2-2ab+b^2)\cos^2\tfrac{1}{2}C \\ &= (a+b)^2\sin^2\tfrac{1}{2}C+(a-b)^2\cos^2\tfrac{1}{2}C \\ &= (a+b)^2\sin^2\tfrac{1}{2}C\left\{1+\left(\frac{a-b}{a+b}\right)^2\cot^2\tfrac{1}{2}C\right\}; \end{aligned}$$

$$\text{or, } = (a-b)^2\cos^2\tfrac{1}{2}C\left\{1+\left(\frac{a+b}{a-b}\right)^2\tan^2\tfrac{1}{2}C\right\}:$$

which are adapted to logarithms by means of a subsidiary angle as in (1): and the preference will be to be decided by the magnitudes of the quantities,

$$a+b,\ a-b,\ \sin\tfrac{1}{2}C,\ \cos\tfrac{1}{2}C;\ \text{and } \frac{a-b}{a+b}\cot\tfrac{1}{2}C,$$

or its reciprocal, agreeably to Article (108).

(4) *Given the three sides* a, b, c: *to find the angles.*

First Solution.

$$\text{Sin } A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}:$$

which is practically convenient for all cases wherein A is not nearly $=90^\circ$. See Articles (73) and (108).

Second Solution.

$$\text{Sin } \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}:$$

which may be conveniently used when A is nearly $=90^\circ$. See Articles (75) and (108).

Third Solution.

$$\text{Cos } \tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}:$$

which may be also used when A is nearly $=90^\circ$: and of these two solutions, the advantage will be with the *former* or the *latter*, according as A is *less* or *greater* han 90°. See Articles (75) and (108).

Fourth Solution.

$$\text{Tan } \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}:$$

which is applicable to all instances in which A is not nearly $=180^\circ$. See Articles (75) and (108).

There is no difficulty in expressing the logarithmic orms at length; and of course, the angles B and C may be found by the requisite changes of symbols.

Examples.

In the right-angled triangle whose sides are 3, 4 nd 5: having given

log 2 = .3010300:	L sin 53°7′ = 9.9030136:
log 3 = .4771213:	L sin 53°8′ = 9.9031084:
log 4 = .6020600:	L tan 53°7′ = 10.1247266:
log 5 = .6989700:	L tan 53°8′ = 10.1249898:

we shall be enabled to solve the following cases, by means of the formulæ above found.

(1) Given $a = 4$, $A = 53^\circ 7' 48''.36$: to find the rest.

Here, $B = 90^\circ - 53^\circ 7' 48''.36 = 36^\circ 52' 11''.64$.

$$\begin{aligned} \text{Log } b &= 10 + \log a - L \tan A \\ &= 10 + \log 4 - L \tan 53^\circ 7' 48''.36 \\ &= \begin{cases} + 10.6020600 \\ - 10.1249387 \end{cases} \\ &= \quad .4771213 = \log 3: \text{ whence, } b = 3. \end{aligned}$$

$$\begin{aligned} \text{Log } c &= 10 + \log a - L \sin A \\ &= 10 + \log 4 - L \sin 53^\circ 7' 48''.36 \\ &= \begin{cases} + 10.6020600 \\ - \ \ 9.9030900 \end{cases} \\ &= \quad .6989700 = \log 5: \text{ whence, } c = 5. \end{aligned}$$

(2) Given $a = 4$, and $b = 3$: to find the rest.

$$\begin{aligned} L \tan A &= 10 + \log a - \log b = 10 + \log 4 - \log 3 \\ &= \begin{cases} + 10.6020600 \\ - \quad .4771213 \end{cases} \\ &= \quad 10.1249387 = L \tan 53^\circ 7' 48''.36: \end{aligned}$$

whence, $A = 53^\circ 7' 48''.36$, and $\therefore B = 36^\circ 52' 11''.64$.

$$\begin{aligned} \text{Log } c &= 10 + \log a - L \sin A \\ &= 10 + \log 4 - L \sin 53^\circ 7' 48''.36 \\ &= \begin{cases} + 10.6020600 \\ - \ \ 9.9030900 \end{cases} \\ &= \quad .6989700 = \log 5: \text{ whence, } c = 5. \end{aligned}$$

By varying the data of this triangle, further practice may be obtained.

(3) In an oblique-angled triangle, having given $a = 320$, $b = 800$ and $B = 128^\circ 4'$: to find the rest;

when $\log\ 2 = .3010300$:

$\log 561 = 2.7489629$:

$\log 562 = 2.7497363$:

$L \sin 18^\circ 21' = 9.4980635$:

$L \sin 18^\circ 22' = 9.4984442$:

$L \sin 33^\circ 34' = 9.7426520$:

$L \sin 33^\circ 35' = 9.7428423$:

$L \sin 51^\circ 56' = 9.8961369$.

$$\begin{aligned} L \sin A &= \log a - \log b + L \sin B \\ &= \log 320 - \log 800 + L \sin 128^\circ 4' \\ &= \left\{\begin{array}{l} +\,2.5051500 \\ +\,9.8961369 \\ -\,2.9030900 \end{array}\right. \\ &= \quad 9.4981969 = L \sin 18^\circ 21' 21'': \end{aligned}$$

$\therefore\ A = 18^\circ 21' 21''$: and $C = 180^\circ - (A + B) = 33^\circ 34' 39''$.

$$\begin{aligned} \text{Log } c &= \log a + L \sin C - L \sin A \\ &= \log 320 + L \sin 33^\circ 34' 39'' - L \sin 18^\circ 21' 21'' \\ &= \left\{\begin{array}{l} +\,2.5051500 \\ +\,9.7427757 \\ -\,9.4981969 \end{array}\right. \\ &= \quad 2.7497288 = \log 561.99, \text{ nearly}: \end{aligned}$$

wherefore, $c = 561.99$, nearly.

Additional practice may be obtained by means of this example, from changing the data.

(4) Given $a = 500$, $b = 400$ and $C = 88^\circ 30'$: to find the rest: when

$\log\ 2 = .3010300$:

$\log\ 3 = .4771213$:

$\log 632 = 2.8007172$:

$\log 633 = 2.8014037$:

$$L \sin 52°15' = 9.8980060:$$
$$L \sin 52°16' = 9.8981038:$$
$$L \sin 88°30' = 9.9998512:$$
$$L \tan\ 6°30' = 9.0566595:$$
$$L \tan\ 6°31' = 9.0577813:$$
$$L \cot 44°15' = 10.0113711.$$

$$L \tan \tfrac{1}{2}(A - B) = \log\left(\frac{a-b}{a+b}\right) + L \cot \tfrac{1}{2}C$$
$$= \begin{cases} +\ 10.0113711 \\ -'\quad .9542426 \end{cases}$$
$$= \qquad 9.0571285 = L \tan 6°30'25'':$$

$$\therefore\ \tfrac{1}{2}(A - B) = 6°30'25'',\ \text{and}\ \tfrac{1}{2}(A + B) = 45°45'0'':$$

whence, $A = 52°15'25''$, and $B = 39°14'35''$.

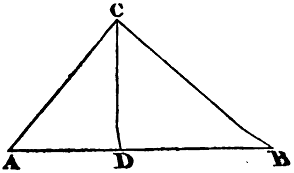
$$\text{Log}\ c = \log a + L \sin C - L \sin A$$
$$= \begin{cases} +\ 2.6989700 \\ +\ 9.9998512 \\ -\ 9.8980467 \end{cases}$$
$$= \qquad 2.8007745 = \log 632.083:$$

whence, $c = 632.083$.

Additional exercises may be had, by varying the data of this example.

112. Though the solutions which have been given include all the cases that can possibly occur, it is proper to observe, that the solutions of *right-angled* triangles may always be made applicable to *oblique-angled* triangles, by drawing a perpendicular from one of the angles upon the opposite side, produced when necessary.

Ex. 1. Given b, c, A; to solve the triangle.



From the angle C draw CD perpendicular to one of e *given* sides AB, produced if necessary: then

$$AD = AC \cos A = b \cos A:$$

$$\therefore\ BD = AB - AD = c - b \cos A:$$

$$\text{also, } CD = AC \sin A = b \sin A:$$

$$\therefore\ \tan B = \frac{CD}{BD} = \frac{b \sin A}{c - b \cos A}:$$

$$\text{whence } \angle\, ACB = 180 - (A + B):$$

$$\text{and } BC = BD \sec B = (c - b \cos A) \sec B.$$

If $\frac{b}{c} \sin A = \tan \theta$, we shall readily obtain

$$\tan B = \sin A \sin \theta \operatorname{cosec} (A - \theta),$$

$$\text{and } BC = c \operatorname{cosec} A \sec B \sin (A - \theta) \sec \theta,$$

which are both adapted to logarithmic computation.

Ex. 2. Given a, b, c: to solve the triangle.

Using the same diagram, we have by Article (72),

$$AB : BC + AC :: BC - AC : BD \mp AD:$$

$$\text{whence } BD \mp AD = \frac{a^2 - b^2}{c}, \text{ is known}:$$

$$\text{also, } BD \pm AD = c, \text{ is given}:$$

nd from these equalities, the values of the segments AD nd BD are immediately obtained: therefore in each f the right-angled triangles CDA, CDB, *two* parts are ow given, and consequently the angles A, B, and thereore the angle C, are readily determined.

This method of solution, avoiding all long logarithnical expressions, is exceedingly simple in practice: nd it is therefore generally adopted by *Land-Surveyors*, *Engineers*, &c.

113. The *Mensuration of Heights, Distances*, &c.: he *Trigonometrical Survey* of a Country: and what is alled *Plane Sailing* in *Navigation*, are merely the appliation *of the Solution of* Triangles to particular instances,

assisted by the use of certain *Mathematical Instruments*, by means of which the lengths of *Lines*, and the magnitudes of *Angles* are ascertained.

Gunter's Chain of 4 *poles* or 22 *yards:* or common *Tape* of 50 or 100 *feet,* is used to measure *Distances:* a *Graduated Quadrant* or *Sextant,* furnished with a *Plumb-Line,* is applied to measure angles of *Elevation* and *Depression* above and below the *Horizontal Line;* a *Theodolite* to measure *Horizontal Angles:* and a *Sextant,* or *Quadrant,* such as are *Oblique.*

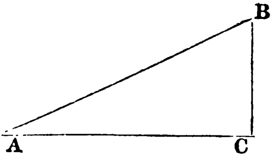
For a particular Description of these Instruments, their *Use* and *Adjustment,* the Reader is referred to *A Treatise on the Principal Mathematical Instruments, by* Mr. F. W. SIMMS.

114. This Application of Trigonometry involves no principles which have not been already explained; and in it no general Rules can be laid down, except that such *Lines* must be *measured,* and such *Angles observed,* as may most easily and conveniently lead to the determination of those which are required.

In a *Trigonometrical Survey* of any considerable extent, certain corrections are required on account of the *Spherical Figure of the Earth,* which will be hereafter explained under the head of *Geodetical Operations;* and at the same time certain contrivances and precautions will be briefly pointed out, in order to ensure the greatest possible degree of accuracy in the results.

Ex. 1. *To find the height of an accessible object standing upon a horizontal plane.*

Let BC be the height of the object standing upon the horizontal plane AC:



measure off $CA = m$ feet, and at A let the angle $CAB = A^{\circ}$, *be observed:* then

$\frac{C}{C} = \tan A$; and $\therefore BC = AC \tan A = m \tan A$, is found.

Also, $\frac{AB}{AC} = \sec A$; and $\therefore AB = AC \sec A = m \sec A$, hich is the distance of the summit of the object from he place of observation, is determined.

As all Instrumental Observations are liable to error, f we suppose the *observed* angle A to become $A + a$, and he corresponding *computed* height H to become $H + h$, e shall have

$$H = m \tan A, \text{ and } H + h = m \tan (A + a):$$

$$\therefore\ h = m \{\tan (A + a) - \tan A\}$$

$$= m \left\{\frac{\sin (A + a)}{\cos (A + a)} - \frac{\sin A}{\cos A}\right\}$$

$$= m \left\{\frac{\sin (A + a) \cos A - \cos (A + a) \sin A}{\cos A \cos (A + a)}\right\}$$

$$= m \left\{\frac{\sin (A + a - A)}{\cos A \cos (A + a)}\right\} = m \frac{\sin a}{\cos A \cos (A + a)}:$$

hich is the value of the error entailed upon the computation of the height H, in consequence of the error a n the observation of the angle A, of elevation.

If a be very small, which is always the case in good nstruments well adjusted, we shall have

$$h = m \frac{\sin a}{\cos^2 A}, \text{ nearly,}$$

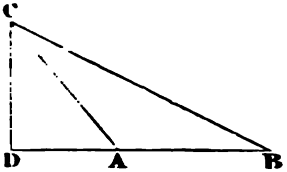
$$= \frac{H}{\tan A} \cdot \frac{\sin a}{\cos^2 A}, \text{ (since } H = m \tan A)$$

$$= \frac{H \sin a}{\sin A \cos A} = \frac{2H \sin a}{\sin 2A}:$$

and this is evidently the least possible when $\sin 2A$ is the greatest possible, or when $2A = 90^\circ$ and $A = 45^\circ$, the magnitude of the error a being supposed small and given: in other words, a small error in the observed angle will *least affect the computed* height, when that angle is 45°: *nd consequently* when the height of the object is equal to the horizontal distance of the place of observation from the foot of it.

Ex. 2. *To find the height of an inaccessible object above a horizontal plane.*

Let the point C denote the place of the object: draw CD perpendicular to the horizontal line BD: and in this line take two positions A, B at a given distance from each other: observe the angles CAD, CBD, which call A and B respectively: then $\angle ACB = A - B$:



$$\text{whence, } \frac{AC}{AB} = \frac{\sin ABC}{\sin ACB} = \frac{\sin B}{\sin (A - B)}:$$

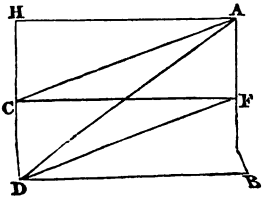
$$\therefore\ AC = AB \frac{\sin B}{\sin (A - B)} \text{ is found:}$$

$$\text{also, } \frac{CD}{AC} = \sin A, \text{ or } CD = AC \sin A:$$

$$\therefore \text{ the height } CD = AB \frac{\sin A \sin B}{\sin (A - B)} \text{ is determined.}$$

This example will enable us to find the distances of any inaccessible object C, from two stations A and B whose distance from each other is known.

Ex. 3. *To determine the height of an object standing upon a horizontal plane, by means of observations made at the top of another object of given height, on the same plane.*



Let the height of the given object $AB=a$: observe e angles of *depression* of the top and bottom of the her object CD, and let them be represented by α, β spectively: draw CF parallel to DB: then

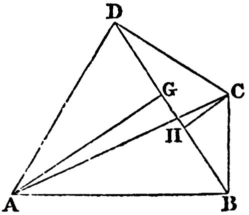
$$\frac{AD}{AB}=\frac{1}{\sin ADB}=\frac{1}{\sin\beta}, \text{ or } AD=\frac{a}{\sin\beta}:$$

$$\frac{CD}{AD}=\frac{\sin CAD}{\sin ACD}=\frac{\sin(\beta-\alpha)}{\sin(90^0+\alpha)}=\frac{\sin(\beta-\alpha)}{\cos\alpha},$$

whence, $CD=\dfrac{a\sin(\beta-\alpha)}{\cos\alpha\sin\beta}$ is the height sought.

Also, $AC=\dfrac{a}{\cos\alpha\tan\beta}$, if its value be required.

Ex. 4. *To find the distance between two objects at* and D, *which are inaccessible to each other, and also to he observer at the stations* A *and* B, *the distance between hich may be measured.*



Let $AB=a$, $\angle BAC=\alpha_1$, $\angle BAD=\alpha_2$:

$\angle ABC=\beta_1$, $\angle ABD=\beta_2$, and $\angle CBD=\gamma$,

ll of which angles may be determined by the proper bservations whether they are in the same plane or not: hen,

$$\text{from } \triangle ABC,\ BC=a\frac{\sin BAC}{\sin BCA}=\frac{a\sin\alpha_1}{\sin(\alpha_1+\beta_1)},$$

which we shall represent by the symbol k:

$$\text{from } \triangle ABD,\ BD=a\frac{\sin BAD}{\sin BDA}=\frac{a\sin\alpha_2}{\sin(\alpha_2+\beta_2)},$$

o be hereafter denoted by l: therefore we have now two sides BC and BD of $\triangle\, BCD$, and the included angle CBD, to find the magnitude of CD: whence,

$$CD^2 = k^2 + l^2 - 2kl \cos CBD$$

$$= k^2 + l^2 - 2kl \cos \gamma :$$

and the adaptation of this result to logarithmic computation has been fully considered in Article (111).

When C and D are in the same plane with A and B, the angle $CBD = ABC - ABD$, or $\gamma = \beta_1 - \beta_2$; and we may suggest, that *many* of the ordinary questions which occur in the mensuration of heights and distances are comprised in this example.

Additional Examples with their solutions will be found in the first Appendix.

CHAPTER V.

THE ANALYSIS OF ANGULAR SECTIONS BY DEMOIVRE'S THEOREM. THE INVESTIGATION OF VARIOUS ANALYTICAL EXPRESSIONS FOR ANGULAR FUNCTIONS, AND THE NUMERICAL VALUE OF THE CIRCUMFERENCE OF THE CIRCLE.

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115. THE preceding chapters comprise what is properly regarded as the *Elements of Trigonometry,* and in them will be found the *Demonstrations* and *Applications* of all the *Principles* and *Theorems* employed in a course of *Elementary* Mathematical Science. We proceed now to a series of propositions established upon *Analytical* grounds, which at the same time that they are indispensably necessary in the *Investigations of Physical Science,* present to our notice many of the results already obtained, as well as several remarkable conclusions which it would be otherwise difficult to arrive at.

116. *To find expressions for all angles, whose sines, cosines, &c. are equal to* sin θ, cos θ, *&c., respectively.*

The reasoning used in (7) of Article (35) gives

$$\sin\theta = \pm\sin(2r\pi \pm \theta) = \pm\sin\{(2r+1)\pi \mp \theta\}:$$

$$\cos\theta = +\cos(2r\pi \pm \theta) = -\cos\{(2r+1)\pi \pm \theta\}:$$

$$\tan\theta = \pm\tan(2r\pi \pm \theta) = \pm\tan\{(2r+1)\pi \pm \theta\}: \ \&c.$$

where r may be any whole number whatever: in the first and last of which, the double sign ±, may be affixed to the angles *themselves* instead of their *functions,* and in the second it may be affixed to the *angles* by leaving the signs of the *functions* as they are, according to Article (31).

Hence, all the functions of the angular magnitude $2r\pi \pm \theta$ will be equal in every respect to those of the angle $\pm\theta$: whereas the functions of $(2r+1)\pi \pm \theta$ *may* differ from them in the algebraical sign: and admitting that θ may be either a *positive* or *negative* quantity, the same circumstances will be found to occur in the following articles.

117. *To find the number of different values comprised in the functions* $\cos\frac{2r\pi+\theta}{n}$ *and* $\sin\frac{2r\pi+\theta}{n}$, *when successive integral values are assigned to* r.

If u with the figures 0, 1, 2, 3, &c. *suffixed,* denote the values of $\cos\frac{2r\pi+\theta}{n}$, corresponding to the values 0, 1, 2, 3, &c. of r respectively, we shall have

$$u_0 = \cos\frac{\theta}{n}: \qquad u_1 = \cos\frac{2\pi+\theta}{n}:$$

$$u_2 = \cos\frac{4\pi+\theta}{n}: \quad u_3 = \cos\frac{6\pi+\theta}{n}: \text{ \&c.}$$

$$u_{n-2} = \cos\frac{(2n-4)\pi+\theta}{n} = \cos\left\{2\pi - \frac{4\pi-\theta}{n}\right\}$$

$$= \cos -\left(\frac{4\pi-\theta}{n}\right) = \cos\frac{4\pi-\theta}{n}:$$

$$u_{n-1} = \cos\frac{(2n-2)\pi+\theta}{n} = \cos\left\{2\pi - \frac{2\pi-\theta}{n}\right\}$$

$$= \cos -\left(\frac{2\pi-\theta}{n}\right) = \cos\frac{2\pi-\theta}{n}:$$

$$u_n = \cos\frac{2n\pi+\theta}{n} = \cos\left(2\pi+\frac{\theta}{n}\right) = \cos\frac{\theta}{n}:$$

$$u_{n+1} = \cos\frac{(2n+2)\pi+\theta}{n} = \cos\left\{2\pi+\frac{2\pi+\theta}{n}\right\}$$

$$= \cos\frac{2\pi+\theta}{n}: \text{ \&c.}:$$

from which it appears there are n, and *only* n different values of $\cos\frac{2r\pi+\theta}{n}$, corresponding to the values 0, 1, 2, &c. $n-1$, of r: inasmuch as the *same values* afterwards *recur* in the *same order:* and the function is therefore said to be *periodical.*

Similarly, for the values of $\sin\frac{2r\pi+\theta}{n}$.

The values above found are in general all *different* from each other: for, if r' and r'' be any two values of r,

$$\cos\left(\frac{2r'\pi+\theta}{n}\right) \text{ cannot be equal to } \cos\left(\frac{2r''\pi+\theta}{n}\right),$$

unless $\frac{2(r''-r')\pi}{n}$ be a multiple of 2π, which is inadmissible, because r', r'' and $\therefore r''-r'<n$: and although *equal* values of the function *may* be obtained by assigning *particular* values to θ, the *number* of its values will still be n. If r be greater than $n-1$, it has been seen that multiples of 2π produce no alteration in the values of the proposed function.

The different values of

$$\cos\left\{\frac{(2r+1)\pi+\theta}{n}\right\}, \text{ and } \sin\left\{\frac{(2r+1)\pi+\theta}{n}\right\},$$

will also be found by putting 0, 1, 2, 3, &c. $n-1$, for r: and if $\theta=0$, the same will therefore hold good for

$$\cos\frac{2r\pi}{n},\ \sin\frac{2r\pi}{n}: \text{ and } \cos\frac{(2r+1)\pi}{n},\ \sin\frac{(2r+1)\pi}{n}:$$

respectively.

DEMOIVRE'S THEOREM.

118. If m be any rational quantity whatever, either integral or fractional, positive or negative:

$$\{\cos\theta\pm\sqrt{-1}\sin\theta\}^m=\cos m\theta\pm\sqrt{-1}\sin m\theta.$$

(1) Let the index be a positive whole number m: then

$$\{\cos\theta\pm\sqrt{-1}\sin\theta\}^2=\{\cos\theta\pm\sqrt{-1}\sin\theta\}$$
$$\times\{\cos\theta\pm\sqrt{-1}\sin\theta\}$$
$$=\cos^2\theta-\sin^2\theta\pm\sqrt{-1}\,2\sin\theta\cos\theta$$
$$=\cos 2\theta\pm\sqrt{-1}\sin 2\theta:$$

$$\{\cos\theta\pm\sqrt{-1}\sin\theta\}^3=\{\cos 2\theta\pm\sqrt{-1}\sin 2\theta\}$$

$$\times \{\cos\theta \pm \sqrt{-1}\sin\theta\}$$
$$= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$$
$$\pm \sqrt{-1}(\sin 2\theta\cos\theta + \cos 2\theta \sin\theta)$$
$$= \cos(2\theta+\theta) \pm \sqrt{-1}\sin(2\theta+\theta)$$
$$= \cos 3\theta \pm \sqrt{-1}\sin 3\theta: \text{ and so on}:$$

hence, if we assume the law observed here, to hold good for the index $m-1$, so that

$$\{\cos\theta \pm \sqrt{-1}\sin\theta\}^{m-1} = \cos(m-1)\theta \pm \sqrt{-1}\sin(m-1)\theta,$$

we shall have

$$\{\cos\theta \pm \sqrt{-1}\sin\theta\}^{m}$$
$$= \{\cos(m-1)\theta \pm \sqrt{-1}\sin(m-1)\theta\}\{\cos\theta \pm \sqrt{-1}\sin\theta\}$$
$$= \cos(m-1)\theta\cos\theta - \sin(m-1)\theta\sin\theta$$
$$\pm \sqrt{-1}\{\sin(m-1)\theta\cos\theta + \cos(m-1)\theta\sin\theta\}$$
$$= \cos m\theta \pm \sqrt{-1}\sin m\theta:$$

or, if the formula be true for any one value of the index, it is proved to be true for the next superior value: but it has been *demonstrated* in the cases when the index is 2 and 3, and therefore it is true when the index is 4, 5, &c.: that is, it is generally true that

$$\{\cos\theta \pm \sqrt{-1}\sin\theta\}^{m} = \cos m\theta \pm \sqrt{-1}\sin m\theta,$$

when the index m is a positive whole number.

(2) Let the index be a negative whole number $-m$: then

$$\{\cos\theta \pm \sqrt{-1}\sin\theta\}^{-m} = \frac{1}{(\cos\theta \pm \sqrt{-1}\sin\theta)^{m}}$$
$$= \frac{1}{(\cos\theta \pm \sqrt{-1}\sin\theta)^{m}} \times \left\{\frac{\cos\theta \mp \sqrt{-1}\sin\theta}{\cos\theta \mp \sqrt{-1}\sin\theta}\right\}^{m}$$
$$= \frac{(\cos\theta \mp \sqrt{-1}\sin\theta)^{m}}{(\cos^2\theta + \sin^2\theta)^{m}} = (\cos\theta \mp \sqrt{-1}\sin\theta)^{m}$$
$$= \cos m\theta \mp \sqrt{-1}\sin m\theta, \text{ by the preceding case,}$$
$$= \cos(-m\theta) \pm \sqrt{-1}\sin(-m\theta), \text{ by Article (31).}$$

(3) Let the index be a fraction, either positive or egative, represented by $\frac{m}{n}$: then,

$$\{\cos\theta \pm \sqrt{-1}\sin\theta\}^m = \cos m\theta \pm \sqrt{-1}\sin m\theta$$
$$= \cos n\left(\frac{m}{n}\theta\right) \pm \sqrt{-1}\sin n\left(\frac{m}{n}\theta\right)$$
$$= \left\{\cos\frac{m}{n}\theta \pm \sqrt{-1}\sin\frac{m}{n}\theta\right\}^n,$$

y the preceding cases;

$$\therefore\ \{\cos\theta \pm \sqrt{-1}\sin\theta\}^{\frac{m}{n}} = \cos\frac{m}{n}\theta \pm \sqrt{-1}\sin\frac{m}{n}\theta.$$

119. Cor. 1. Hence, when m is a positive whole iumber, by Article (24), we shall have

$$(\cos\theta \pm \sqrt{-1}\sin\theta)^m$$
$$= \{\cos(2r\pi + \theta) \pm \sqrt{-1}\sin(2r\pi + \theta)\}^m$$
$$= \cos(2mr\pi + m\theta) \pm \sqrt{-1}\sin(2mr\pi + m\theta)$$
$$= \cos m\theta \pm \sqrt{-1}\sin m\theta:$$

hat is, each member of the equality has only one value, vhatever whole number m may be.

If however the index be fractional, as $\frac{m}{n}$; then will

$$\{\cos\theta \pm \sqrt{-1}\sin\theta\}^{\frac{m}{n}}$$
$$= \{\cos(2r\pi + \theta) \pm \sqrt{-1}\sin(2r\pi + \theta)\}^{\frac{m}{n}}$$
$$= \cos\frac{m}{n}(2r\pi + \theta) \pm \sqrt{-1}\sin\frac{m}{n}(2r\pi + \theta),$$

vhereof the latter member has n different values corresponding to the values 0, 1, 2, &c. $n - 1$, of r, as seen n Article (117): and these n values are the n roots of he expression $\{\cos\theta \pm \sqrt{-1}\sin\theta\}^m$: that is, each member of the equality has the same number of different alues, which are merely *indicated* in the former, but *nay be actually exhibited* in the latter, by assigning to r *e values 0, 1, 2, 3, &c.* $n - 1$.

120. Cor. 2. We may express the Theorem in a form better adapted for use, when the index is fractional, as follows.

For, $(\cos\theta \pm \sqrt{-1}\sin\theta)^m = \cos m\theta \pm \sqrt{-1}\sin m\theta$

$$= \cos(2r\pi + m\theta) \pm \sqrt{-1}\sin(2r\pi + m\theta):$$

$$\therefore\ (\cos\theta \pm \sqrt{-1}\sin\theta)^{\frac{m}{n}}$$

$$= \{\cos(2r\pi + m\theta) \pm \sqrt{-1}\sin(2r\pi + m\theta)\}^{\frac{1}{n}}$$

$$= \cos\frac{1}{n}(2r\pi + \pm m\theta)\sqrt{-1}\sin\frac{1}{n}(2r\pi + m\theta):$$

from which the n values are obtained, by assigning to r, the values 0, 1, 2, 3, &c. $n-1$, as before.

From this it appears that $(\cos\theta \pm \sqrt{-1}\sin\theta)^{\frac{m}{n}}$ is in all respects equivalent to either

$$\sqrt[n]{(\cos\theta \pm \sqrt{-1}\sin\theta)^m}, \text{ or } \{\sqrt[n]{\cos\theta \pm \sqrt{-1}\sin\theta}\}^m,$$

as in common algebraical quantities.

121. Cor. 3. Hence the *Algebraical* binomial $a + b\sqrt{-1}$, as well as its *powers* and *roots*, may be expressed by means of *Trigonometrical* functions.

For, if $a + b\sqrt{-1} = p(\cos\theta + \sqrt{-1}\sin\theta)$,

we have $p\cos\theta = a$, and $p\sin\theta = b$;

$$\therefore\ \tan\theta = \frac{p\sin\theta}{p\cos\theta} = \frac{b}{a}, \text{ which gives immediately}$$

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}},\quad \sin\theta = \frac{b}{\sqrt{a^2+b^2}},\quad \text{and } \therefore\ p = \sqrt{a^2+b^2}:$$

that is, $a + b\sqrt{-1} = \sqrt{a^2+b^2}(\cos\theta + \sqrt{-1}\sin\theta)$:

where θ is known from any of the equalities above.

Hence, $(a + b\sqrt{-1})^{\frac{1}{n}} = (a^2+b^2)^{\frac{1}{2n}}(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{1}{n}}$

$$= (a^2+b^2)^{\frac{1}{2n}}\{\cos\frac{1}{n}(2r\pi + \theta) + \sqrt{-1}\sin\frac{1}{n}(2r\pi + \theta)\}.$$

Also, $(a - b\sqrt{-1})^{\frac{1}{n}} = (a^2 + b^2)^{\frac{1}{2n}}(\cos\theta - \sqrt{-1}\sin\theta)^{\frac{1}{n}}$

$$= (a^2 + b^2)^{\frac{1}{2n}}\left\{\cos\frac{1}{n}(2r\pi + \theta) - \sqrt{-1}\sin\frac{1}{n}(2r\pi + \theta)\right\}:$$

whence, by addition and subtraction, we obtain

$$(a + b\sqrt{-1})^{\frac{1}{n}} + (a - b\sqrt{-1})^{\frac{1}{n}}$$

$$= 2(a^2 + b^2)^{\frac{1}{2n}}\cos\frac{1}{n}(2r\pi + \theta), \text{ which is } \textit{real}:$$

$$\text{and } (a + b\sqrt{-1})^{\frac{1}{n}} - (a - b\sqrt{-1})^{\frac{1}{n}}$$

$$= 2(a^2 + b^2)^{\frac{1}{2n}}\sqrt{-1}\sin\frac{1}{n}(2r\pi + \theta), \text{ which is } \textit{imaginary}.$$

If the index be $\frac{m}{n}$, the process is precisely the same, and the results possess the same character.

Ex. To find the *three* cube roots of $3 + 4\sqrt{-1}$,

we have $\cos\theta = \frac{3}{5}$, $\sin\theta = \frac{4}{5}$, and $p = \sqrt{a^2 + b^2} = 5$:

the admissible values of r being 0, 1, 2: whence, the three cube roots required will be

$$\sqrt[3]{5}\left\{\cos\frac{\theta}{3} + \sqrt{-1}\sin\frac{\theta}{3}\right\}:$$

$$\sqrt[3]{5}\left\{\cos\left(\frac{2\pi + \theta}{3}\right) + \sqrt{-1}\sin\left(\frac{2\pi + \theta}{3}\right)\right\}:$$

$$\sqrt[3]{5}\left\{\cos\left(\frac{4\pi + \theta}{3}\right) + \sqrt{-1}\sin\left(\frac{4\pi + \theta}{3}\right)\right\}:$$

which are all known by means of a table of Natural Cosines, since $\cos\theta = \frac{3}{5} = .6 = \cos 53^\circ 8'$, nearly.

122. *To express the sines and cosines of θ and* mθ *in the forms of algebraical binomials;* m *being either integral or fractional, positive or negative.*

Since, $1 = \cos^2\theta + \sin^2\theta$

$$= (\cos\theta + \sqrt{-1}\sin\theta)(\cos\theta - \sqrt{-1}\sin\theta):$$

if we assume $\cos\theta + \sqrt{-1}\sin\theta = x$, we shall have

$$\cos\theta - \sqrt{-1}\sin\theta = \frac{1}{x}:$$

∴ by addition and subtraction, we obtain

$$2\cos\theta = x + \frac{1}{x}, \text{ and } 2\sqrt{-1}\sin\theta = x - \frac{1}{x}.$$

Also, $\cos m\theta + \sqrt{-1}\sin m\theta = (\cos\theta + \sqrt{-1}\sin\theta)^m = x^m$,

and $\cos m\theta - \sqrt{-1}\sin m\theta = (\cos\theta - \sqrt{-1}\sin\theta)^m = \frac{1}{x^m}$:

by *Demoivre's* Theorem: whence, as above,

$$2\cos m\theta = x^m + \frac{1}{x^m}, \text{ and } 2\sqrt{-1}\sin m\theta = x^m - \frac{1}{x^m}.$$

If the index be fractional, we have by Article (119),

$$2\cos\frac{m}{n}(2r\pi + \theta) = x^{\frac{m}{n}} + \frac{1}{x^{\frac{m}{n}}}:$$

$$2\sqrt{-1}\sin\frac{m}{n}(2r\pi + \theta) = x^{\frac{m}{n}} - \frac{1}{x^{\frac{m}{n}}}:$$

where the symbol r admits of the values 0, 1, 2, &c $n-1$, so that each of the members has the same numbe n, of different values.

123. *To express the sine and cosine of a multipl angle $m\theta$, in terms of the sine and cosine of the simpl angle θ.*

Here, $\cos m\theta + \sqrt{-1}\sin m\theta = \{\cos\theta + \sqrt{-1}\sin\theta\}^m$

$$= \cos^m\theta + m\cos^{m-1}\theta\sqrt{-1}\sin\theta - \frac{m(m-1)}{1.2}\cos^{m-2}\theta\sin^2\theta$$
$$- \frac{m(m-1)(m-2)}{1.2.3}\cos^{m-3}\theta\sqrt{-1}\sin^3\theta + \&c.$$

by the Binomial Theorem.

Whence, equating the real and imaginary parts both members of this equality, we have

$$\cos m\theta = \cos^m\theta - \frac{m(m-1)}{1.2}\cos^{m-2}\theta\sin^2\theta + \&c.:$$

ich will be continued to $\frac{1}{2}(m+1)$ terms, if m be odd,
d to $(\frac{1}{2}m+1)$ terms, if m be even: also,

$$m\theta = m\cos^{m-1}\theta\sin\theta - \frac{m(m-1)(m-2)}{1.2.3}\cos^{m-3}\theta\sin^3\theta$$
$$+\ \&c.:$$

ich will be continued to $\frac{1}{2}(m+1)$ terms, if m be odd,
d to $(\frac{1}{2}m)$ terms, if m be even.

124. Cor. 1. The last term of the expanded bino-
ial above used, is evidently $(-1)^{\frac{1}{2}m}\sin^m\theta$, and the last
t one is $(-1)^{\frac{1}{2}(m-1)}m\cos\theta\sin^{m-1}\theta$.

(1) If m be *odd* and of the *form* $4p+1$, the last
rms in the expressions for $\sin m\theta$ and $\cos m\theta$ will be
n$^{m}\theta$ and $m\cos\theta\sin^{m-1}\theta$ respectively: but if it be of the
m $4p+3$, they will be $-\sin^m\theta$ and $-m\cos\theta\sin^{m-1}\theta$.

(2) If m be *even* and of the *form* $4p$, the last terms
ll be $-m\cos\theta\sin^{m-1}\theta$ and $\sin^m\theta$ respectively: but if
be of the *form* $4p+2$, they will be $m\cos\theta\sin^{m-1}\theta$
d $-\sin^m\theta$.

Examples.

$\text{Sin}\, 2\theta = 2\cos\theta\sin\theta$:
$\cos 2\theta = \cos^2\theta - \sin^2\theta$.
$\text{Sin}\, 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$:
$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$.
$\text{Sin}\, 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$:
$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$.
$\text{Sin}\, 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$:
$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$.

125. Cor. 2. Tan $m\theta$ in terms of $\tan\theta$, may be
ınd from

$$\tan m\theta = \frac{\sin m\theta}{\cos m\theta}:$$

t we have immediately, $\tan m\theta$

$$\frac{1}{\sqrt{-1}}\left\{\frac{(\cos m\theta + \sqrt{-1}\sin m\theta) - (\cos m\theta - \sqrt{-1}\sin m\theta)}{(\cos m\theta + \sqrt{-1}\sin m\theta) + (\cos m\theta - \sqrt{-1}\sin m\theta)}\right\}$$

$$= \frac{1}{\sqrt{-1}} \left\{ \frac{(\cos\theta + \sqrt{-1}\sin\theta)^m - (\cos\theta - \sqrt{-1}\sin\theta)^m}{(\cos\theta + \sqrt{-1}\sin\theta)^m + (\cos\theta - \sqrt{-1}\sin\theta)^m} \right\}$$

$$= \frac{1}{\sqrt{-1}} \left\{ \frac{(1 + \sqrt{-1}\tan\theta)^m - (1 - \sqrt{-1}\tan\theta)^m}{(1 + \sqrt{-1}\tan\theta)^m + (1 - \sqrt{-1}\tan\theta)^m} \right\}$$

$$= \frac{m\tan\theta - \frac{m(m-1)(m-2)}{1.2.3}\tan^3\theta + \&c.}{1 - \frac{m(m-1)}{1.2}\tan^2\theta + \&c.}:$$

where the last terms of the numerator and denominato will be analogous to those pointed out in Article (66).

Examples.

$$\text{Tan } 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

$$\text{Tan } 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

$$\text{Tan } 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

$$\text{Tan } 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}.$$

126. *To express the positive integral powers of cosine of an angle, in terms of the cosines of its multipl*

Assuming $2\cos\theta = x + \frac{1}{x}$, as in Article (122), we h

$$2^m \cos^m\theta = \left(x + \frac{1}{x}\right)^m$$

$$= x^m + mx^{m-2} + \frac{m(m-1)}{1.2}x^{m-4} + \&c.$$

$$+ \frac{m(m-1)}{1.2}\frac{1}{x^{m-4}} + m\frac{1}{x^{m-2}} + \frac{1}{x^m}$$

$$= \left(x^m + \frac{1}{x^m}\right) + m\left(x^{m-2} + \frac{1}{x^{m-2}}\right) + \frac{m(m-1)}{1.2}\left(x^{m-4} + \frac{1}{x^{m-4}}\right) +$$

placing together the terms *equidistant* from the inning and the end.

(1) If m be *odd*, the number of terms of the ex- ded binomial will be $m+1$, which is *even*, and there l therefore be an exact number $\frac{1}{2}(m+1)$ of pairs of ns, formed as above: whence we have

$$2^m \cos^m \theta = 2 \cos m\theta + 2m \cos (m-2)\theta$$

$$+ 2 \frac{m(m-1)}{1.2} \cos (m-4)\theta + \&c. \text{ to } \tfrac{1}{2}(m+1) \text{ terms.}$$

(2) If m be *even*, the number of terms of the ex- ded binomial being $m+1$, will be *odd*, so that besides pairs of terms as above, there will be an additional which is the $(\frac{1}{2}m+1)^{\text{th}}$, and does not involve x, since $^m = x^0 = 1$: and this being

$$\frac{m(m-1)(m-2)\,\&c.\,(\frac{1}{2}m+1)}{1.2.3.\&c.\,\frac{1}{2}m},$$

ich (see the Algebra) is easily reducible to

$$\frac{1.3.5.\&c.\,(m-1)}{1.\,2.3.\&c.\,\frac{1}{2}m}\, 2^{\frac{1}{2}m}:$$

have $2^m \cos^m \theta = 2 \cos m\theta + 2m \cos (m-2)\theta$

$$+ 2 \frac{m(m-1)}{1.2} \cos (m-4)\theta + \&c. \text{ to } \tfrac{1}{2}m \text{ terms}$$

$$+ \frac{1.3.5.\&c.\,(m-1)}{1.2.3.\&c.\,\frac{1}{2}m}\, 2^{\frac{1}{2}m}.$$

In the first case, the last term of the expansion in its ter form, is evidently the sum of the two middle terms the expanded binomial, which are the $\frac{1}{2}(m+1)^{\text{th}}$ and $n+3)^{\text{th}}$, involving x and $\frac{1}{x}$ respectively, and it will refore be $2 \frac{m(m-1)(m-2)\,\&c.\,\frac{1}{2}(m+3)}{1.2.3.\&c.\,\frac{1}{2}(m-1)} \cos \theta$.

Examples.

$2 \cos^2\theta = \cos 2\theta + 1.$

$2^2 \cos^3\theta = \cos 3\theta + 3 \cos \theta.$

$2^3 \cos^4\theta = \cos 4\theta + 4 \cos 2\theta + 3.$

$2^4 \cos^5\theta = \cos 5\theta + 5 \cos 3\theta + 10 \cos \theta.$

127. *To express the positive integral powers of the sine of an angle, in terms of the sines or cosines of the multiple angles.*

If, from Article (122), we assume $2\sqrt{-1}\sin\theta = x - \frac{1}{x}$:

$$\text{then, } 2^m(-1)^{\frac{m}{2}}\sin^m\theta = \left(x - \frac{1}{x}\right)^m$$

$$x^m - mx^{m-2} + \frac{m(m-1)}{1.2}x^{m-4} - \&c.$$

$$\pm\frac{m(m-1)}{1.2}\frac{1}{x^{m-4}} \mp m\frac{1}{x^{m-2}} \pm \frac{1}{x^m},$$

wherein the *lower* or *upper* sign must be used, according as m is *odd* or *even*, as is shewn in the *Algebra.*

(1). If m be *odd,* it must be of one of the *forms* $4p+1$ and $4p+3$:

$$\text{also, } (-1)^{\frac{4p+1}{2}} = (-1)^{2p+\frac{1}{2}} = (-1)^{\frac{1}{2}} = \sqrt{-1}:$$

$$\text{and } (-1)^{\frac{4p+3}{2}} = (-1)^{2p+\frac{3}{2}} = (-1)^{\frac{3}{2}} = -\sqrt{-1}:$$

$$\therefore \pm 2^m\sqrt{-1}\sin^m\theta = \left(x^m - \frac{1}{x^m}\right) - m\left(x^{m-2} - \frac{1}{x^{m-2}}\right)$$

$$+\frac{m(m-1)}{1.2}\left(x^{m-4} - \frac{1}{x^{m-4}}\right) - \&c. \text{ to } \tfrac{1}{2}(m+1) \text{ terms,}$$

where the upper sign corresponds to the *form* $4p+1$ of m, and the lower to the *form* $4p+3$: that is,

$$\pm 2^m\sqrt{-1}\sin^m\theta = 2\sqrt{-1}\sin m\theta - 2m\sqrt{-1}\sin(m-2)\theta$$

$$+2\frac{m(m-1)}{1.2}\sqrt{-1}\sin(m-4)\theta - \&c. \text{ to } \tfrac{1}{2}(m+1) \text{ terms:}$$

and by dividing both members by $\sqrt{-1}$, we have

$$\pm 2^m\sin^m\theta = 2\sin m\theta - 2m\sin(m-2)\theta$$

$$+2\frac{m(m-1)}{1.2}\sin(m-4)\theta - \&c. \text{ to } \tfrac{1}{2}(m+1) \text{ terms.}$$

(2) If m be *even*, it must be of one of the *forms* $4p$ and $4p+2$:

$$\text{but } (-1)^{\frac{4p}{2}} = (-1)^{2p} = 1:$$

$$\text{and } (-1)^{\frac{4p+2}{2}} = (-1)^{2p+1} = -1:$$

$$\therefore \text{ we have } \pm 2^m \sin^m \theta = \left(x^m + \frac{1}{x^m}\right) - m\left(x^{m-2} + \frac{1}{x^{m-2}}\right)$$

$$+ \frac{m(m-1)}{1 \cdot 2}\left(x^{m-4} + \frac{1}{x^{m-4}}\right) - \&\text{c. to } \tfrac{1}{2}m \text{ terms}$$

$$\pm \frac{1 \cdot 3 \cdot 5 \cdot \&\text{c. } (m-1)}{1 \cdot 2 \cdot 3 \cdot \&\text{c. } \frac{1}{2}m} 2^{\frac{1}{2}m},$$

e last term being the middle one as before, and the pper and lower signs corresponding to the *forms* $4p$ nd $4p + 2$ of m respectively: that is,

$$\pm 2^m \sin^m \theta = 2 \cos m\theta - 2m \cos (m - 2)\theta$$

$$+ 2\frac{m(m-1)}{1 \cdot 2} \cos (m - 4)\theta - \&\text{c. to } \tfrac{1}{2}m \text{ terms}$$

$$\pm \frac{1 \cdot 3 \cdot 5 \cdot \&\text{c. } (m-1)}{1 \cdot 2 \cdot 3 \cdot \&\text{c. } \frac{1}{2}m} 2^{\frac{1}{2}m}.$$

Examples.

$2 \sin^2\theta = 1 - \cos 2\theta.$

$2^2 \sin^3\theta = 3 \sin\theta - \sin 3\theta.$

$2^3 \sin^4\theta = 3 - 4 \cos 2\theta + \cos 4\theta.$

$2^4 \sin^5\theta = 10 \sin \theta - 5 \sin 3\theta + \sin 5\theta.$

128. *To express the sine and cosine of an angle, n terms of the angle itself.*

In Article (123), it has been proved that

$$\sin m\theta = m \cos^{m-1}\theta \sin \theta - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} \cos^{m-3}\theta \sin^3\theta$$

$$+ \&\text{c.}$$

$$= \cos^m \theta \left\{m \tan \theta - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} \tan^3\theta + \&\text{c.}\right\}:$$

$$\text{and, } \cos m\theta = \cos^m \theta - \frac{m(m-1)}{1 \cdot 2} \cos^{m-2}\theta \sin^2\theta + \&\text{c.}$$

$$= \cos^m\theta \left\{1 - \frac{m(m-1)}{1 \cdot 2} \tan^2 \theta + \&\text{c.}\right\};$$

whence, assuming $m\theta = a$, or $m = \frac{a}{\theta}$, we have

$$\sin a = \cos^m\theta \left\{\frac{a}{\theta}\tan\theta - \frac{\frac{a}{\theta}\left(\frac{a}{\theta}-1\right)\left(\frac{a}{\theta}-2\right)}{1.2.3}\tan^3\theta + \&c.\right\}$$

$$= \cos^m\theta \left\{a\left(\frac{\tan\theta}{\theta}\right) - \frac{a(a-\theta)(a-2\theta)}{1.2.3}\left(\frac{\tan\theta}{\theta}\right)^3 + \&c.\right\}:$$

$$\text{and, } \cos a = \cos^m\theta \left\{1 - \frac{\frac{a}{\theta}\left(\frac{a}{\theta}-1\right)}{1.2}\tan^2\theta + \&c.\right\}$$

$$= \cos^m\theta \left\{1 - \frac{a(a-\theta)}{1.2}\left(\frac{\tan\theta}{\theta}\right)^2\right.$$

$$\left. + \frac{a(a-\theta)(a-2\theta)(a-3\theta)}{1.2.3.4}\left(\frac{\tan\theta}{\theta}\right)^4 - \&c.\right\}:$$

and these equalities being true independently of any *particular* value of m, they must remain so, when $m = \infty$, or $\theta = 0$: but by Article (86), we have *ultimately*,

$$\cos\theta = 1,\ \cos^m\theta = 1, \text{ and } \frac{\tan\theta}{\theta} = 1:$$

$$\therefore \sin a = a - \frac{a^3}{1.2.3} + \frac{a^5}{1.2.3.4.5} - \&c.:$$

$$\cos a = 1 - \frac{a^2}{1.2} + \frac{a^4}{1.2.3.4} - \&c.:$$

and they must both be supposed to be continued *in infinitum,* because $m = \infty$.

Since in this article, $m = \frac{a}{\theta}$, and $\frac{\tan\theta}{\theta}$ *ultimately* $= 1$:

it follows from Article (87) that a is expressed in the circular measure, or that it is the arc of a circle whose radius is 1: and the sine and cosine of an *angle* A° will be found, by substituting for a in the latter members, the fraction $\frac{A^0}{r^0}$, where $r^0 = 57^\circ.2957795$ &c. by Article *(13).*

Ex. If $A = 1''$, we have $a = .00000484813$ &c. by Article (15):

$$\therefore \sin 1'' = .00000484813 \text{ \&c.} - \frac{1}{1.2.3}(.00000484813 \text{ \&c.})^3 + \text{\&c.}$$

$$= .00000484813 \text{ \&c., } \textit{very nearly.}$$

These two equalities, being purely *Symbolical* or *Analytical*, will, like other expressions of the same kind, be of little or no use for *Arithmetical Computation*, except when the series *converge* with a degree of rapidity suficient to render a few of the terms a good approximation o the true value: they are still however of the greatest mportance in *Analysis*, as will be seen hereafter; and whenever we meet with their latter members as the results of an investigation, we conclude with certainty hat they adequately express sin a and cos a respectively.

129. Cor. 1. When a is very small, as for instance, when a is the circular measure of $1''$, we have $\sin a = a$, or $\sin 1'' = 1''$ nearly; therefore if θ denote the circular measure of any other angle A, we have

$$\frac{\text{the number of seconds in } A}{1''} = \frac{\text{arc } \theta}{\text{arc } 1''} = \frac{\theta}{\sin 1''}:$$

$\therefore$ the number of seconds in the angle A whose circular measure is θ, will be expressed by

$$\frac{\theta}{\sin 1''}.$$

This equation enables us to avoid the trouble of first inding the number of degrees, &c. corresponding to θ, and afterwards reducing them to seconds: and it is of great practical utility when θ is expressed in terms of angular functions, inasmuch as the number of seconds is nearly obtained from multiplying it by the invariable factor

$$\frac{1}{\sin 1''} = \frac{1}{.00000484813 \text{ \&c.}} = 206265, \text{ nearly.}$$

Thus, if $\theta = 1$, the value of the corresponding angle n seconds $= 206265$, *nearly, which differs very slightly from the true value, as* appears by Article (15).

130. Cor. 2. By substitution, we shall find

$$\tan a = \frac{\sin a}{\cos a} = a + \frac{a^3}{1 \,.\, 3} + \frac{2a^5}{1 \,.\, 3 \,.\, 5} + \frac{17a^7}{1 \,.\, 3 \,.\, 3 \,.\, 5 \,.\, 7} + \&c.:$$

$$\text{chd}\, a = 2 \sin \tfrac{1}{2} a = a - \frac{a^3}{1 \,.\, 3 \,.\, 2^3} + \frac{a^5}{1 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 2^5} - \&c.:$$

wherefore if a be small compared to the radius of the circle of which it is an arc, we have

$$2 \sin a + \tan a = 2a - \frac{2a^3}{1 \,.\, 2 \,.\, 3} + a + \frac{2a^3}{1 \,.\, 2 \,.\, 3} = 3a, \text{ nearly}:$$

$$8 \,\text{chd} \frac{a}{2} - \text{chd}\, a = 8 \left\{ \frac{a}{2} - \frac{a^3}{1 \,.\, 3 \,.\, 4^3} \right\} - \left\{ a - \frac{a^3}{1 \,.\, 3 \,.\, 2^3} \right\}$$

$$= 4a - \frac{a^3}{1 \,.\, 3 \,.\, 2^3} - a + \frac{a^3}{1 \,.\, 3 \,.\, 2^3} = 3a, \text{ nearly}:$$

which are two useful approximate formulæ for *practice*, found in most Books of Mensuration.

131. *To express the sine and cosine of an angle, by means of imaginary exponential functions of the angle itself.*

In the expression $e^x = 1 + \frac{x}{1} + \frac{x^2}{1 \,.\, 2} + \frac{x^3}{1 \,.\, 2 \,.\, 3} + \&c.$

where $e =$ the numerical magnitude 2.71828 &c., let $a\sqrt{-1}$ and $-a\sqrt{-1}$ be successively substituted in the place of x, and we shall have

$$e^{a\sqrt{-1}} = 1 + \frac{a\sqrt{-1}}{1} - \frac{a^2}{1 \,.\, 2} - \frac{a^3\sqrt{-1}}{1 \,.\, 2 \,.\, 3} + \frac{a^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \&c.$$

$$e^{-a\sqrt{-1}} = 1 - \frac{a\sqrt{-1}}{1} - \frac{a^2}{1 \,.\, 2} + \frac{a^3\sqrt{-1}}{1 \,.\, 2 \,.\, 3} + \frac{a^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} - \&c.:$$

whence, by addition and subtraction, we get

$$e^{a\sqrt{-1}} + e^{-a\sqrt{-1}} = 2 \left\{ 1 - \frac{a^2}{1 \,.\, 2} + \frac{a^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} - \&c. \right\}$$

$$= 2 \cos a:$$

$$e^{a\sqrt{-1}} - e^{-a\sqrt{-1}} = 2\sqrt{-1} \left\{ a - \frac{a^3}{1.2.3} + \frac{a^5}{1.2.3.4.5} - \&c. \right\}$$

$$= 2\sqrt{-1} \sin a:$$

$$\text{that is, } \cos a = \frac{1}{2}\left\{e^{a\sqrt{-1}} + e^{-a\sqrt{-1}}\right\},$$

$$\text{and } \sin a = \frac{1}{2\sqrt{-1}}\left\{e^{a\sqrt{-1}} - e^{-a\sqrt{-1}}\right\}.$$

These equalities, which are imaginary in *form* only, ising entirely from the generalizations of *Symbolical lgebra* cannot immediately furnish any *Arithmetical* sults; but the truth of the Principles of that Science ill evidently justify their use, whenever they can be plied as *instruments* of investigation: and we need erefore entertain no doubt as to the accuracy of the nclusions to which they lead.

132. Cor. Hence, we have immediately

$$\tan a = \frac{\sin a}{\cos a} = \frac{1}{\sqrt{-1}}\left\{\frac{e^{a\sqrt{-1}} - e^{-a\sqrt{-1}}}{e^{a\sqrt{-1}} + e^{-a\sqrt{-1}}}\right\}$$

$$= \frac{1}{\sqrt{-1}}\left\{\frac{e^{2a\sqrt{-1}} - 1}{e^{2a\sqrt{-1}} + 1}\right\},$$

y multiplying the numerator and denominator by $e^{a\sqrt{-1}}$.

These formulæ, which are reckoned amongst the ost important discoveries of the *Modern Analysis*, are ue to *Euler*.

133. *To express the length of a circular arc, in terms f its tangent.*

From the formulæ just investigated, we have

$$^{2a\sqrt{-1}} = \frac{e^{a\sqrt{-1}}}{e^{-a\sqrt{-1}}} = \frac{\cos a + \sqrt{-1}\sin a}{\cos a - \sqrt{-1}\sin a} = \frac{1 + \sqrt{-1}\tan a}{1 - \sqrt{-1}\tan a}:$$

hence, if the Napierian logarithms of both members e taken, and the ordinary expansions be used:

$$2a\sqrt{-1} = \log_e(1 + \sqrt{-1}\tan a) - \log_e(1 - \sqrt{-1}\tan a)$$

$$= \sqrt{-1}\tan a + \tfrac{1}{2}\tan^2 a - \tfrac{1}{3}\sqrt{-1}\tan^3 a - \tfrac{1}{4}\tan^4 a + \&c.$$

$$+ \sqrt{-1}\tan a - \tfrac{1}{2}\tan^2 a - \tfrac{1}{3}\sqrt{-1}\tan^3 a + \tfrac{1}{4}\tan^4 a + \&c.$$

and equating the terms on each side affected with the symbol $\sqrt{-1}$, we have

$$a = \tan a - \tfrac{1}{3}\tan^3 a + \tfrac{1}{5}\tan^5 a - \&c.:$$

which evidently converges in all cases where tan a is not greater than 1.

Without employing the *Logarithmic Series*,

$$\log_e y = (y-1) - \tfrac{1}{2}(y-1)^2 + \tfrac{1}{3}(y-1)^3 - \&c.$$

here used, which will be deduced in the *seventh* chapter, we have, by means of Article (125),

$$\frac{\tan ma}{m} = \frac{\tan a - \dfrac{(m-1)(m-2)}{1\,.\,2\,.\,3}\tan^3 a + \&c.}{1 - \dfrac{m(m-1)}{1\,.\,2}\tan^2 a + \&c.},$$

which is true for *all* values of m: whence, if $m = 0$,

$$\text{and } \therefore \frac{\tan ma}{m} = \frac{ma}{m} = a,$$

by Article (86), we conclude immediately that

$$a = \tan a - \tfrac{1}{3}\tan^3 a + \tfrac{1}{5}\tan^5 a - \&c.: \text{ as before.}$$

This remarkable series is due to *Mr James Gregory*, who was eminently distinguished for the discoveries he made in the various departments of *mathematical science*, was born at *Aberdeen* in the year 1639, and Professor of Mathematics in the College of *St Andrew;* and it seems that his process was somewhat analogous to the one last given.

134. Cor. 1. If a be the circular measure of

$$30^\circ = \tfrac{1}{6}\pi, \text{ then } \tan a = \frac{1}{\sqrt{3}}:$$

$$\therefore \tfrac{1}{6}\pi = \frac{1}{\sqrt{3}}\left\{1 - \frac{1}{3\,.\,3} + \frac{1}{5\,.\,3^2} - \frac{1}{7\,.\,3^3} + \frac{1}{9\,.\,3^4} - \&c.\right\}:$$

which is a series whose rate of convergence is such, that it would require a great number of terms to be used in order to compute the circumference of the circle with accuracy, to many places of figures.

The invention of this series is due to *Dr Edmund Halley*, a most eminent Mathematician and Philosopher, was born in *London* in 1656: and became suc- ively *Professor of Geometry* at *Oxford*, and *Astronomer* *al* at *Greenwich.*

By means of *Halley's* series above given, *Mr Abraham* *rp*, of *Little Horton*, near *Bradford*, *Yorkshire*, who born about the year 1651, calculated the value of circumference of a circle to 72 places of decimals. *John Machin*, *Professor of Astronomy* at *Gresham* *lege*, about the year 1713 extended it to 100 places decimals by the same series; and finally, *M. de Lagny*, eminent French Mathematician, born at *Lyons* about 0, continued it to 128 places of decimals.

135. Cor. 2. If $\alpha = 45^\circ$, and $\therefore \tan \alpha = 1$: we have

$$\begin{aligned}\tfrac{1}{4}\pi &= 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \&c.\\ &= (1 - \tfrac{1}{3}) + (\tfrac{1}{5} - \tfrac{1}{7}) + (\tfrac{1}{9} - \tfrac{1}{11}) + \&c.\\ &= \frac{2}{1\,.\,3} + \frac{2}{5\,.\,7} + \frac{2}{9\,.\,11} + \&c.:\end{aligned}$$

t this series from the slow rate of its convergence, uld require a great number of terms to be taken, to ure a close approximation to the true value of $\frac{1}{4}\pi$.

136. Cor. 3. Let α and β be two arcs whose tan- its are $\frac{1}{2}$ and $\frac{1}{3}$ respectively: then

$$= \frac{1}{2} - \frac{1}{3\,.\,2^3} + \frac{1}{5\,.\,2^5} - \&c.: \ \beta = \frac{1}{3} - \frac{1}{3\,.\,3^3} + \frac{1}{5\,.\,3^5} - \&c.:$$

$$\therefore\ \alpha + \beta = \left(\frac{1}{2} + \frac{1}{3}\right) - \tfrac{1}{3}\left(\frac{1}{2^3} + \frac{1}{3^3}\right) + \tfrac{1}{5}\left(\frac{1}{2^5} + \frac{1}{3^5}\right) - \&c.:$$

$$\text{but, } \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2}\,.\,\frac{1}{3}}$$

$$= 1 = \tan \frac{\pi}{4}: \text{ and } \therefore \alpha + \beta = \frac{\pi}{4}:$$

$$\text{ence, } \tfrac{1}{4}\pi = \left(\frac{1}{2} + \frac{1}{3}\right) - \tfrac{1}{3}\left(\frac{1}{2^3} + \frac{1}{3^3}\right) + \tfrac{1}{5}\left(\frac{1}{2^5} + \frac{1}{3^5}\right) - \&c.:$$

ch is given by *Euler* in his *Introductio in Analysin* *nitorum*, and converges with so considerable a degree apidity that a small number of terms will give a good roximation, *but an objection to it is*, the alternation of *signs of its terms.*

137. Cor. 4. If $\tan \alpha = \frac{1}{5}$, we have $\tan 2\alpha = \frac{5}{12}$,

$$\therefore \tan 4\alpha = \frac{120}{119} = 1\frac{1}{119}, \text{ which is somewhat} > \tan\frac{\pi}{4}:$$

whence, if $y = 4\alpha - \frac{\pi}{4}$, we find $\tan y = \frac{\tan 4\alpha - 1}{\tan 4\alpha + 1} = \frac{1}{239}$:

$$\text{but } 4\alpha = 4\left\{\frac{1}{5} - \frac{1}{3 \,.\, 5^3} + \frac{1}{5 \,.\, 5^5} - \&c.\right\}:$$

$$\text{and } y = \left\{\frac{1}{239} - \frac{1}{3 \,.\, 239^3} + \frac{1}{5 \,.\, 239^5} - \&c.\right\}:$$

$$\therefore \frac{\pi}{4} = 4\left\{\frac{1}{5} - \frac{1}{3 \,.\, 5^3} + \frac{1}{5 \,.\, 5^5} - \&c.\right\}$$

$$-\left\{\frac{1}{239} - \frac{1}{3 \,.\, 239^3} + \frac{1}{5 \,.\, 239^5} - \&c.\right\}.$$

The convergence of both the series here given is very considerable: if *three* terms of the first and *one* of the second be taken, we find $\pi = 3 \,.\, 1416$, which is the value commonly used: but if we take *eight* terms of the first and *three* of the second, we shall have

$$\pi = 3 \,.\, 141592653589793 \ \&c.$$

which is accurate to the last place of decimals.

This series is the discovery of *Mr John Machin* above mentioned, and it seems probable that the laborious computation, referred to in Article (134), was made before his own series had occurred to him.

This subject will be resumed in the first Appendix, where some remarkable results will be deduced with great facility, by means of a peculiar notation for *inverse* trigonometrical functions.

138. The preceding Articles contain a succinct view of the subject as expressed at the head of this chapter, and nothing but the ordinary *Algebraical Theorems* have been made use of, in order to arrive at the results which we have obtained. The complete discussion will however require the assistance of the *Differential Calculus*, and the *student* is referred to Recherches sur *l'Analyse des Sections Angulaires par* M. Poinsot.

139. We shall conclude this chapter with the follow-
g propositions, which are frequently met with, and if
ey be entitled to claim a place at all in an elementary
ork, seem to be admissible here.

140. *Given the sines and cosines of any number of
gles, to find the sine and cosine of their sum.*

From Article (63), we have immediately

$$\cos(\alpha+\beta+\&c.+\lambda)+\sqrt{-1}\sin(\alpha+\beta+\&c.+\lambda)$$
$$=\cos\alpha\cos\beta\,\&c.\cos\lambda$$

ultiplied into

$$+\sqrt{-1}\tan\alpha)(1+\sqrt{-1}\tan\beta)\,\&c.\,(1+\sqrt{-1}\tan\lambda):$$

hence, using the notation adopted in Article (65), the
rmer member

$$\cos\alpha\cos\beta\,\&c.\cos\lambda\,\{1+\sqrt{-1}s_1-s_2-\sqrt{-1}s_3+s_4+\&c.\}:$$
$$\cos(\alpha+\beta+\&c.+\lambda)=\cos\alpha\cos\beta\,\&c.\cos\lambda\,\{1-s_2+s_4-\&c.\}:$$
$$n(\alpha+\beta+\&c.+\lambda)=\cos\alpha\cos\beta\,\&c.\cos\lambda\,\{s_1-s_3+s_5-\&c.\}.$$

From these, we easily prove that if $c_0=\cos\alpha\cos\beta\,\&c.$
m factors, and $c_m=$ the sum of the products of all the
sines *but m*, multiplied by the sines of those *m*,

$$\cos(\alpha+\beta+\&c.+\lambda)=c_0-c_2+c_4-\&c.:$$
$$\sin(\alpha+\beta+\&c.+\lambda)=c_1-c_3+c_5-\&c.$$

Hence also, $\tan(\alpha+\beta+\&c.+\lambda)=\dfrac{s_1-s_3+s_5-\&c.}{1-s_2+s_4-\&c.}$,

ich has been considered in Article (65); and all the
ults here obtained may be verified for *two* or *three*
gles by what has been done in the second chapter.

141. *Given two sides and the included angle of a
ngle, to find the remaining side and angles, by means
infinite series.*

By Article (131), $c^2=a^2+b^2-2ab\cos C$

$$=a^2+b^2-ab\,\{e^{C\sqrt{-1}}+e^{-C\sqrt{-1}}\}$$
$$=(a-be^{C\sqrt{-1}})\,(a-be^{-C\sqrt{-1}}):$$

whence, taking the Napierian logarithms, we have

$$2\log_e c = \log_e(a - be^{C\sqrt{-1}}) + \log_e(a - be^{-C\sqrt{-1}})$$

$$= 2\log_e a + \log_e\left(1 - \frac{b}{a}e^{C\sqrt{-1}}\right) + \log_e\left(1 - \frac{b}{a}e^{-C\sqrt{-1}}\right)$$

$$= 2\log_e a - \frac{b}{a}\left\{e^{C\sqrt{-1}} + e^{-C\sqrt{-1}}\right\}$$

$$- \frac{b^2}{2a^2}\left\{e^{2C\sqrt{-1}} + e^{-2C\sqrt{-1}}\right\}$$

$$- \frac{b^3}{3a^3}\left\{e^{3C\sqrt{-1}} + e^{-3C\sqrt{-1}}\right\} - \&c.:$$

$$\therefore \log_e c = \log_e a - \frac{b}{a}\cos C - \frac{b^2}{2a^2}\cos 2C - \frac{b^3}{3a^3}\cos 3C - \&c$$

Also, since $\frac{a}{b} = \frac{\sin A}{\sin B} = \frac{\sin(B+C)}{\sin B} = \cos C + \frac{\sin C}{\tan B}$,

we have $\tan B = \frac{b\sin C}{a - b\cos C}$, as in Ex. 1 of Article (112)

$$\therefore \frac{e^{2B\sqrt{-1}} - 1}{e^{2B\sqrt{-1}} + 1} = \frac{b\,(e^{C\sqrt{-1}} - e^{-C\sqrt{-1}})}{2a - b\,(e^{C\sqrt{-1}} + e^{-C\sqrt{-1}})}:$$

whence, $e^{2B\sqrt{-1}} = \frac{a - be^{-C\sqrt{-1}}}{a - be^{C\sqrt{-1}}} = \frac{1 - \frac{b}{a}e^{-C\sqrt{-1}}}{1 - \frac{b}{a}e^{C\sqrt{-1}}}$:

and taking the Napierian logarithms, we find

$$B = \frac{b}{a}\left\{\frac{e^{C\sqrt{-1}} - e^{-C\sqrt{-1}}}{2\sqrt{-1}}\right\} + \frac{b^2}{2a^2}\left\{\frac{e^{2C\sqrt{-1}} - e^{-2C\sqrt{-1}}}{2\sqrt{-1}}\right\}$$

$$+ \frac{b^3}{3a^3}\left\{\frac{e^{3C\sqrt{-1}} - e^{-3C\sqrt{-1}}}{2\sqrt{-1}}\right\} + \&c.$$

$$= \frac{b}{a}\sin C + \frac{b^2}{2a^2}\sin 2C + \frac{b^3}{3a^3}\sin 3C + \&c.:$$

and the value of B expressed in seconds will therefo by Article (129), be found from the expression

$$B = \frac{b}{a}\,\frac{\sin C}{\sin 1''} + \frac{b^2}{2a^2}\,\frac{\sin 2C}{\sin 1''} + \frac{b^3}{3a^3}\,\frac{\sin 3C}{\sin 1''} + \&c.$$

It need scarcely be observed that these results will of no manner of use, unless b be very small comared with a; but when this is the case, a few terms the series will give good approximations to their ue values.

142. *To solve triangles without the assistance of athematical Tables.*

In a right-angled triangle, we have $a = c \sin A$: hence if $r^0 = 57^0.2957795$ &c., the circular measure of e angle A is $\frac{A^0}{r^0}$: and therefore by Article (128), we find

$$= c\left\{\left(\frac{A^0}{r^0}\right) - \frac{1}{1.2.3}\left(\frac{A^0}{r^0}\right)^3 + \frac{1}{1.2.3.4.5}\left(\frac{A^0}{r^0}\right)^5 - \&c.\right\}.$$

Also, $\tan A = \frac{a}{b}$: and by Article (133), we obtain

$$A^0 = r^0\left\{\left(\frac{a}{b}\right) - \tfrac{1}{3}\left(\frac{a}{b}\right)^3 + \tfrac{1}{5}\left(\frac{a}{b}\right)^5 - \&c.\right\}.$$

In an oblique-angled triangle, we have

$$c^2 = a^2 + b^2 - 2ab \cos C = a^2 + b^2 - 2ab\sqrt{1 - \sin^2 C}$$

$$= a^2 + b^2 - 2ab\sqrt{1 - \left\{\left(\frac{C^0}{r^0}\right) - \frac{1}{1.2.3}\left(\frac{C^0}{r^0}\right)^3 + \&c.\right\}^2}:$$

om which c is determined.

All these solutions are of course merely approximaons to the true values: the *first* and *third* will be ractically useful only when the angles A and C are mall compared to the angular unit of circular measure $7^0.2957795$ &c.: and the *second*, when the side a opposite he required angle, is very small compared to the side which is adjacent to it.

Various other instances of this kind may be seen in *onnycastle's* Trigonometry.

143. *When two angles of a triangle are very acute, find approximate values of its different parts.*

(1) Let the two very acute angles A, B and the ide c be given: then we have

$$\sin A = A - \frac{A^3}{1.2.3}: \ \sin B = B - \frac{B^3}{1.2.3}:$$

$$\text{and } \sin C = \sin(A+B) = (A+B) - \frac{(A+B)^3}{1 \,.\, 2 \,.\, 3}, \text{ nearly:}$$

$$\text{hence, } a = \frac{c \sin A}{\sin(A+B)} = \frac{cA}{A+B}\left\{1 + \frac{2AB+B^2}{1 \,.\, 2 \,.\, 3}\right\}:$$

$$\text{and } b = \frac{c \sin B}{\sin(A+B)} = \frac{cB}{A+B}\left\{1 + \frac{2AB+A^2}{1 \,.\, 2 \,.\, 3}\right\}, \text{ nearly,}$$

by neglecting the terms involving the higher power of A and B.

Hence, also, $a + b - c = \frac{1}{2} ABc$.

(2) Let two sides a, b and the included angle C which is very obtuse, be given: then if $C = 180^0 - \alpha$, α is very small, and we have

$$c^2 = a^2 + b^2 - 2ab \cos C = a^2 + b^2 + 2ab \cos \alpha$$

$$= a^2 + b^2 + 2ab\left(1 - \frac{\alpha^2}{1 \,.\, 2}\right), \text{ nearly,}$$

$$= (a+b)^2 - ab\alpha^2, \text{ nearly:}$$

$$\therefore\ c = \sqrt{(a+b)^2 - ab\alpha^2} = a + b - \tfrac{1}{2}\frac{ab\alpha^2}{a+b}, \text{ nearly.}$$

$$\text{Sin } A = \frac{a}{c}\sin C = \frac{a}{c}\sin\alpha = \frac{a}{c}\left\{\alpha - \frac{\alpha^3}{1 \,.\, 2 \,.\, 3}\right\}, \text{ nearly:}$$

$$\text{but } \frac{a}{c} = \frac{a}{a + b - \frac{1}{2}\frac{ab\alpha^2}{a+b}}$$

$$= \frac{a}{a+b}\left\{\frac{1}{1 - \frac{1}{2}\frac{ab\alpha^2}{(a+b)^2}}\right\} = \frac{a}{a+b}\left\{1 - \tfrac{1}{2}\frac{ab\alpha^2}{(a+b)^2}\right\}^{-1}$$

$$= \frac{a}{a+b}\left\{1 + \tfrac{1}{2}\frac{ab\alpha^2}{(a+b)^2}\right\}, \text{ nearly;}$$

$$\therefore\ \sin A = \frac{a}{a+b}\left\{1 + \tfrac{1}{2}\frac{ab\alpha^2}{(a+b)^2}\right\}\left(\alpha - \frac{\alpha^3}{1 \,.\, 2 \,.\, 3}\right)$$

$$= \frac{a\alpha}{a+b}\left\{1 + \frac{ab\alpha^2}{2(a+b)^2} - \frac{\alpha^2}{1 \,.\, 2 \,.\, 3}\right\}$$

$$= \frac{a\alpha}{a+b}\left\{1 - \frac{a^2 - ab + b^2}{(a+b)^2}\,\frac{\alpha^2}{1 \,.\, 2 \,.\, 3}\right\}, \text{ nearly}$$

also, $\sin A = A - \frac{A^3}{1.2.3} = A - \frac{\sin^3 A}{1.2.3}$, nearly:

$$\therefore\ A = \sin A + \frac{\sin^3 A}{1.2.3}$$

$$= \frac{a\alpha}{a+b}\left\{1 - \frac{a^2 - ab + b^2}{(a+b)^2}\,\frac{\alpha^2}{1.2.3}\right\} + \left(\frac{a}{a+b}\right)^3 \frac{\alpha^3}{1.2.3}$$

$$= \frac{a\alpha}{a+b}\left\{1 - \frac{a^2 - ab + b^2 - a^2}{(a+b)^2}\,\frac{\alpha^2}{1.2.3}\right\}$$

$$= \frac{a\alpha}{a+b}\left\{1 + \frac{b(a-b)}{(a+b)^2}\,\frac{\alpha^2}{1.2.3}\right\}, \text{ nearly.}$$

Similarly, $B = \frac{b\alpha}{a+b}\left\{1 - \frac{a(a-b)}{(a+b)^2}\,\frac{\alpha^2}{1.2.3}\right\}$, nearly.

It is of course understood here, that the angles are expressed in the circular measure: and it is immediately seen that $A + B = \alpha$.

CHAPTER VI.

THE APPLICATION OF TRIGONOMETRY TO THE SOLUTION OF EQUATIONS, AND THE RESOLUTION OF CERTAIN EXPRESSIONS INTO THEIR SIMPLE AND QUADRATIC FACTORS.

I. *Quadratic Equations.*

144. Let the proposed equation be $x^2-px+q=0$: then, by the ordinary method, we have

$$x=\frac{p\pm\sqrt{p^2-4q}}{2}=\frac{p}{2}\left\{1\pm\sqrt{1-\frac{4q}{p^2}}\right\}:$$

whence, if the roots be real, and therefore $\frac{4q}{p^2}$ be not greater than 1, and we assume $\sin^2\theta=\frac{4q}{p^2}$: then will

$$x=\frac{p}{2}\{1\pm\sqrt{1-\sin^2\theta}\}=\frac{p}{2}\{1\pm\cos\theta\}:$$

but $1+\cos\theta=2\cos^2\frac{1}{2}\theta$, and $1-\cos\theta=2\sin^2\frac{1}{2}\theta$:

$\therefore$ the values of x are $p\cos^2\frac{\theta}{2}$ and $p\sin^2\frac{\theta}{2}$.

These values may also be exhibited in another form

for since $p=\frac{2\sqrt{q}}{\sin\theta}$, we have

$$x=p\cos^2\tfrac{1}{2}\theta=\frac{2\sqrt{q}}{\sin\theta}\cos^2\tfrac{1}{2}\theta=\sqrt{q}\cot\tfrac{1}{2}\theta:$$

$$x=p\sin^2\tfrac{1}{2}\theta=\frac{2\sqrt{q}}{\sin\theta}\sin^2\tfrac{1}{2}\theta=\sqrt{q}\tan\tfrac{1}{2}\theta.$$

If the proposed equation be $x^2+px+q=0$,

then will $x=-p\cos^2\frac{1}{2}\theta=-\sqrt{q}\cot\frac{1}{2}\theta$:

and $x=-p\sin^2\frac{1}{2}\theta=-\sqrt{q}\tan\frac{1}{2}\theta$:

hich differ from the former only in their algebraical gns.

In Logarithms: if x_1 and x_2 be the two values of x, e shall have

$$L \sin\theta = \tfrac{1}{2}\{20 + 2\log 2 + \log q - 2\log p\},$$

hich determines θ: also,

$$\log(\pm x_1) = \log p + 2\, L \cos\tfrac{1}{2}\theta - 20:$$
$$\log(\pm x_2) = \log p + 2\, L \sin\tfrac{1}{2}\theta - 20.$$

145. Let the proposed equation be $x^2 - px - q = 0$: hen, as before,

$$x = \frac{p \pm \sqrt{p^2 + 4q}}{2} = \frac{p}{2}\left\{1 \pm \sqrt{1 + \frac{4q}{p^2}}\right\}:$$

hence, assuming $\tan^2\theta = \dfrac{4q}{p^2}$, we have

$$x = \frac{p}{2}\{1 \pm \sqrt{1 + \tan^2\theta}\} = \frac{p}{2}\{1 \pm \sec\theta\}:$$

$$\text{but } 1 + \sec\theta = \tan\theta \cot\tfrac{1}{2}\theta:$$
$$\text{and } 1 - \sec\theta = -\tan\theta \tan\tfrac{1}{2}\theta:$$

$\therefore$ the values of x are $\dfrac{p}{2}\tan\theta\cot\tfrac{1}{2}\theta$ and $-\dfrac{p}{2}\tan\theta\tan\tfrac{1}{2}\theta$.

By substituting for p its equal $\dfrac{2\sqrt{q}}{\tan\theta}$, we have the alues of x represented by $\sqrt{q}\cot\tfrac{1}{2}\theta$ and $-\sqrt{q}\tan\tfrac{1}{2}\theta$.

If the proposed equation be $x^2 + px - q = 0$:

$$\text{then will } x = -\frac{p}{2}\tan\theta\cot\tfrac{1}{2}\theta = -\sqrt{q}\cot\tfrac{1}{2}\theta:$$

$$\text{and } x = \frac{p}{2}\tan\theta\tan\tfrac{1}{2}\theta = \sqrt{q}\tan\tfrac{1}{2}\theta:$$

hich differ only in their signs from the former.

In Logarithms: if the roots be distinguished as efore, we shall have

$$L \tan\theta = \tfrac{1}{2}\{20 + 2\log 2 + \log q - 2\log p\}:$$
$$\log(\pm x_1) = \tfrac{1}{2}\log q + L\cot\tfrac{1}{2}\theta - 10:$$
$$\log(\pm x_2) = \tfrac{1}{2}\log q + L\tan\tfrac{1}{2}\theta - 10.$$

146. Cor. In the first case, when $\frac{4q}{p^2}$ is greater than 1, the roots are imaginary: and if $\sec^2\theta = \frac{4q}{p^2}$, they will be expressed by $\frac{p}{2}\{1 \pm \tan\theta\sqrt{-1}\}$: and it may be observed that these solutions can be advantageously employed in those cases only, wherein p and q are very large numbers, or very complicated irrational quantities.

II. *Cubic Equations.*

147. Let the second term of the equation be taken away, when necessary, so that it may be of the form

$$x^3 - qx - r = 0:$$

then, since $\cos\theta$ is not greater than 1, if $x = m\cos\theta$,

we have $m^3\cos^3\theta - mq\cos\theta - r = 0$:

$$\text{or, } \cos^3\theta - \frac{q}{m^2}\cos\theta - \frac{r}{m^3} = 0:$$

but, from $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, in Article (47),

we have $\cos^3\theta - \frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta = 0$:

and these two equations being compared give

$$\frac{q}{m^2} = \frac{3}{4}, \text{ and } \therefore m = 2\sqrt{\frac{q}{3}}:$$

$$\frac{r}{m^3} = \frac{1}{4}\cos 3\theta, \text{ and } \therefore \cos 3\theta = \frac{r}{2}\sqrt{\frac{27}{q^3}}:$$

whence $\cos 3\theta$ is given, and therefore 3θ may be found from the tables, and thus θ becomes known: also, since

$$\cos 3\theta = \cos(2\pi + 3\theta) = \cos(2\pi - 3\theta) = \&c.,$$

the three values of $x = m\cos\theta$ will, by Article (117), be

$$2\sqrt{\frac{q}{3}}\cos\theta,\ 2\sqrt{\frac{q}{3}}\cos\left(\frac{2\pi + 3\theta}{3}\right),\ 2\sqrt{\frac{q}{3}}\cos\left(\frac{2\pi - 3\theta}{3}\right).$$

Ex. 1. Let the equation be $x^3 - 3x - 1 = 0$: then $m = 2$, and $\cos 3\theta = \frac{1}{2} = \cos 60^\circ$, or $\theta = 20^\circ$: from which the values of x are found to be $2\cos 20^\circ$, $-2\cos 40^\circ$

nd $-2\cos 80^\circ$, all of which may be numerically determined by the tables of natural cosines: and are 1.8793852, -1.5320888 and $-.3472964$ respectively.

Ex. 2. Given $x^3-6x-4=0$, to find the values of x.

Here $m=2\sqrt{2}$, and $\cos 3\theta=\dfrac{1}{\sqrt{2}}=\cos 45^\circ$, or $\theta=15$:

whence the three values of $x=m\cos\theta$ will be

$$2\sqrt{2}\cos 15^\circ,\ -2\sqrt{2}\cos 45^\circ \text{ and } -2\sqrt{2}\cos 75^\circ.$$

In this example, the numerical values of the roots re easily found: for we have seen that

$$\cos 15^\circ=\frac{\sqrt{3}+1}{2\sqrt{2}},\ \text{and}\ \cos 75^\circ=\frac{\sqrt{3}-1}{2\sqrt{2}}:$$

nd therefore the roots of the proposed equation are

$$2\sqrt{2}\cos 15^\circ=1+\sqrt{3}:\ -2\sqrt{2}\cos 45^\circ=-2:$$

$$\text{and } -2\sqrt{2}\cos 75^\circ=1-\sqrt{3}:$$

nd the equation is equivalent to, or identical with

$$(x+2)(x-1-\sqrt{3})(x-1+\sqrt{3})=0.$$

148. Cor. Since $\cos 3\theta$ cannot be greater than 1,

$$\frac{r}{2}\sqrt{\frac{27}{q^3}} \text{ cannot be greater than } 1:$$

$$\therefore\ \frac{r}{2}\sqrt{\frac{27}{q^3}}\times\sqrt{\frac{q^3}{27}} \text{ cannot be greater than } \sqrt{\frac{q^3}{27}}:$$

$$\text{or, } \frac{r^2}{4} \text{ cannot be greater than } \frac{q^3}{27}:$$

hat is, $\dfrac{r^2}{4}-\dfrac{q^3}{27}$ cannot be a positive quantity: and this circumstance corresponds with the *irreducible case* of *Cardan's* Solution, all the roots being real: and therefore shews that the solution is available whenever *Cardan's* fails, but not otherwise. See the *Algebra.*

149. Cor. 2. If $\cos 3\theta$ be nearly equal to ± 1, the angle 3θ cannot be accurately determined from the tables, as *appears from Article* (108), and therefore little depen-

dence can be placed upon the correctness of the solution. In this case $\frac{r^2}{4}$ is nearly equal to $\frac{q^3}{27}$, and consequently two roots of the equation are nearly equal to each other, as is evident from the Formula of *Cardan.*

150. Cor. 3. If r be negative, or the equation be $x^3 - qx + r = 0$, the same solution holds good: but it will generally be found more convenient for this form, to assume $x = m \sin\theta$, instead of $x = m\cos\theta$.

151. Cor. 4. If q be negative, or the equation be $x^3 + qx - r = 0$, the values of m and $\cos 3\theta$ are both imaginary: but the following process will give the real root:

$$\text{For, } m = 2\sqrt{\frac{q}{3}}\sqrt{-1}, \text{ and } \cos 3\theta = \frac{r}{2}\sqrt{\frac{27}{q^3}}\sqrt{-1}:$$

$$\therefore\ m^3 \cos 3\theta = 4r, \text{ and } m^3\sqrt{-1}\sin 3\theta = 8\sqrt{\frac{r^2}{4} + \frac{q^3}{27}}:$$

whence, by addition and subtraction, we find

$$m^3(\cos 3\theta \pm \sqrt{-1}\sin 3\theta) = 8\left\{\frac{r}{2} \pm \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right\}:$$

$$\therefore\ m(\cos\theta + \sqrt{-1}\sin\theta) = 2\sqrt[3]{\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}}:$$

$$m(\cos\theta - \sqrt{-1}\sin\theta) = 2\sqrt[3]{\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}}:$$

$$\text{and } x = m\cos\theta = \sqrt[3]{\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}} + \sqrt[3]{\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}};$$

which has *one real* and *two imaginary* values, corresponding to the three cube roots of 1.

This result is the Solution of *Cardan*: and only two roots are imaginary, though it might at first sight be supposed, from the forms of m and $\cos 3\theta$, that they were all imaginary.

152. Cor. 5. If the equation be $x^3 + qx + r = 0$,

$$\text{and } \tan^2\alpha = \frac{4q^3}{27r^2}, \text{ or } \frac{r^2}{4} = \frac{q^3}{27\tan^2\alpha}.$$

'e shall have, $x = \left(-\frac{r}{2}\right)^{\frac{1}{3}}\{(1+\sec\alpha)^{\frac{1}{3}} + (1-\sec\alpha)^{\frac{1}{3}}\}$

$$= \left(\frac{q}{3}\right)^{\frac{1}{2}}\left\{\left(\frac{\sec\alpha+1}{\tan\alpha}\right)^{\frac{1}{3}} - \left(\frac{\sec\alpha-1}{\tan\alpha}\right)^{\frac{1}{3}}\right\}$$

$$= \left(\frac{q}{3}\right)^{\frac{1}{2}}\{(\cot\tfrac{1}{2}\alpha)^{\frac{1}{3}} - (\tan\tfrac{1}{2}\alpha)^{\frac{1}{3}}\}:$$

nce, taking $(\cot\frac{1}{2}\alpha)^{\frac{1}{3}} = \cot\beta$, and $\therefore$ $(\tan\frac{1}{2}\alpha)^{\frac{1}{3}} = \tan\beta$,

$$x = \left(\frac{q}{3}\right)^{\frac{1}{2}}\{\cot\beta - \tan\beta\} = 2\left(\frac{q}{3}\right)^{\frac{1}{2}}\cot 2\beta,$$

h is the real root in a form adapted to logarithms.

Vhen the equation is $x^3 - qx + r = 0$, and $27r^2 > 4q^3$,

by assuming $\sin^2\alpha = \frac{4q^3}{27r^2}$, or $\frac{r^2}{4} = \frac{q^3}{27\sin^2\alpha}$,

ave

$$\left(-\frac{r}{2}\right)^{\frac{1}{3}}\{(1-\cos\alpha)^{\frac{1}{3}} + (1+\cos\alpha)^{\frac{1}{3}}\} = \left(\frac{q}{3}\right)^{\frac{1}{2}}\left\{\left(\frac{1-\cos\alpha}{\sin\alpha}\right)^{\frac{1}{3}}\right.$$

$$\left. + \left(\frac{1+\cos\alpha}{\sin\alpha}\right)^{\frac{1}{3}}\right\} = \left(\frac{q}{3}\right)^{\frac{1}{2}}\{(\tan\tfrac{1}{2}\alpha)^{\frac{1}{3}} + (\cot\tfrac{1}{2}\alpha)^{\frac{1}{3}}\}$$

$\left(\frac{q}{3}\right)^{\frac{1}{2}}\{\tan\beta + \cot\beta\} = 2\left(\frac{q}{3}\right)^{\frac{1}{2}}\operatorname{cosec} 2\beta$, the real root.

aking $\cos 3\theta = \frac{a}{\sqrt{a^2+b^2}}$, and $\therefore$ $\sin 3\theta = \frac{b}{\sqrt{a^2+b^2}}$:

'e $a = -\frac{1}{2}r$, and $b\sqrt{-1} = \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}$, and substi-
g in the formula of *Cardan*, the solution of Article
), will be obtained.

III. *Binomial Equations.*

53. *To find the roots of the equation* $x^n - 1 = 0$: *and* *olve* $x^n - 1$ *into its simple and quadratic factors.*

'ere, $x^n = 1 = \cos 2r\pi + \sqrt{-1}\sin 2r\pi$;

whence, by *Demoivre's* Theorem, we have

$$x = \{\cos 2r\pi + \sqrt{-1}\sin 2r\pi\}^{\frac{1}{n}} = \cos\frac{2r\pi}{n} + \sqrt{-1}\sin\frac{2r\pi}{n};$$

whereof the n different values are found by assigning to r the values 0, 1, 2, 3, &c, $n-1$, by Article (117):

$$\text{but, } \cos\frac{2(n-1)\pi}{n} = \cos\left(2\pi - \frac{2\pi}{n}\right) = \cos\frac{2\pi}{n}:$$

$$\sin\frac{2(n-1)\pi}{n} = \sin\left(2\pi - \frac{2\pi}{n}\right) = -\sin\frac{2\pi}{n}:$$

$$\cos\frac{2(n-2)\pi}{n} = \cos\left(2\pi - \frac{4\pi}{n}\right) = \cos\frac{4\pi}{n}:$$

$$\sin\frac{2(n-2)\pi}{n} = \sin\left(2\pi - \frac{4\pi}{n}\right) = -\sin\frac{4\pi}{n}: \&c.:$$

from which it appears that the values of x obtained by assigning to r the values 1, 2, 3, &c. are the same as those obtained by making r successively equal to $n-1$, $n-2$, $n-3$, &c., except that the signs of the imaginary terms are different: whence the general formula for x is

$$x = \cos\frac{2r\pi}{n} \pm \sqrt{-1}\sin\frac{2r\pi}{n}:$$

and the values of x will be *real*, only when the latter term becomes $= 0$: that is, when $\frac{2r\pi}{n}$ is equal to 0, or to some multiple of π.

(1) Let n be *odd*, or of the *forms* $2m \pm 1$; then the only *real* root is $x = 1$, corresponding to $r = 0$: and the rest being imaginary, occur in pairs, and are determined from the expression just given by assigning to r, the successive values, 1, 2, 3, &c, $\frac{1}{2}(n-1)$.

(2) Let n be *even*, or of the *form* $2m$; then, corresponding to the values 0 and $\frac{1}{2}n$ of r, the *real* values of x are 1 and -1: and the rest, being imaginary, will be found from

$$x = \cos\frac{2r\pi}{n} \pm \sqrt{-1}\sin\frac{2r\pi}{n},$$

by assigning to r, the values 1, 2, 3, &c, $\frac{1}{2}(n-2)$.

154. Cor. Hence, if n be *odd*, we shall have $x^n - 1$ xpressed by means of its simple and quadratic factors: hus,

$$^n - 1 = (x-1)\left(x^2 - 2\cos\frac{2\pi}{n}x + 1\right)\left(x^2 - 2\cos\frac{4\pi}{n}x + 1\right)$$

$$\&c.\ldots\left(x^2 - 2\cos\frac{(n-1)\pi}{n}x + 1\right),$$

ıere being one *simple* factor, and $\frac{1}{2}(n-1)$ *quadratic* fac›rs formed by combining respectively the simple factors, hich differ only in the signs of their imaginary parts.

If n be *even*, we have likewise

$$^n - 1 = (x^2-1)\left(x^2 - 2\cos\frac{2\pi}{n}x + 1\right)\left(x^2 - 2\cos\frac{4\pi}{n}x + 1\right)$$

$$\&c.\ldots\left(x^2 - 2\cos\frac{(n-2)\pi}{n}x + 1\right),$$

onsisting of $\frac{1}{2}n$ *quadratic* factors, the first arising from he combination of the two simple real factors $x - 1$ and $+ 1$, and the rest from the imaginary factors as before.

These two conclusions are immediate consequences f the *Theory of Equations.*

155. *To find the roots of the equation* $x^n + 1 = 0$: *nd to resolve* $x^n + 1$ *into its simple and quadratic factors.*

Here, $x^n = -1 = \cos(2r+1)\pi + \sqrt{-1}\sin(2r+1)\pi$: rom which, by *Demoivre's* Theorem, we have

$$x = \{\cos(2r+1)\pi + \sqrt{-1}\sin(2r+1)\pi\}^{\frac{1}{n}}$$

$$= \cos\frac{(2r+1)\pi}{n} + \sqrt{-1}\sin\frac{(2r+1)\pi}{n}:$$

ınd the n values of x will be obtained, by giving to r, he values 0, 1, 2, 3, &c, $n-1$, by Article (117);

$$\text{ut, }\cos\frac{\{2(n-1)+1\}\pi}{n} = \cos\left(2\pi - \frac{\pi}{n}\right) = \cos\frac{\pi}{n}:$$

$$\sin\frac{\{2(n-1)+1\}\pi}{n} = \sin\left(2\pi - \frac{\pi}{n}\right) = -\sin\frac{\pi}{n}:$$

$$\cos\frac{\{2(n-2)+1\}\pi}{n} = \cos\left(2\pi - \frac{3\pi}{n}\right) = \cos\frac{3\pi}{n}:$$

$$\sin\frac{\{2(n-2)+1\}\pi}{n}=\sin\left(2\pi-\frac{3\pi}{n}\right)=-\sin\frac{3\pi}{n}:\ \&c:$$

whence it appears that the values of x arising from the values 0, 1, 2, 3, &c. of r, are the same respectively as those which arise from the values, $n-1$, $n-2$, $n-3$, &c. of r, except that the signs of the latter terms are different: and the values of x will then be given by

$$x=\cos\frac{(2r+1)\pi}{n}\pm\sqrt{-1}\,\sin\frac{(2r+1)\pi}{n}:$$

and whenever the latter term becomes $=0$, the corresponding values of x will be *real*, as in the former case.

(1) Let n be *odd;* then corresponding to

$r=\frac{1}{2}(n-1)$, we have $x=-1$,

the only real root: and the rest being imaginary are found from the expression above given by assigning to r, the values 0, 1, 2, 3, &c, $\frac{1}{2}(n-3)$.

(2) Let n be *even:* then there is no real root, since $\frac{(2r+1)\pi}{n}$ cannot be 0, or a multiple of π: and the imaginary roots are obtained from the general expression for x, by giving to r the values 0, 1, 2, 3, &c, $\frac{1}{2}(n-2)$.

156. Cor. As in the preceding case, we have, when n is *odd:*

$$x^n+1=(x+1)\left(x^2-2\cos\frac{\pi}{n}x+1\right)\left(x^2-2\cos\frac{3\pi}{n}x+1\right)$$
$$\&c.\dots\left(x^2-2\cos\frac{(n-2)\pi}{n}x+1\right):$$

and when n is *even:*

$$x^n+1=\left(x^2-2\cos\frac{\pi}{n}x+1\right)\left(x^2-2\cos\frac{3\pi}{n}x+1\right)$$
$$\&c.\dots\left(x^2-2\cos\frac{(n-1)\pi}{n}x+1\right).$$

157. The roots of the equations $x^n\mp1=0$, which, by *Demoivre's* Theorem, may be made to appear in order as the terms of *geometrical progressions*, are the values of *all the* n^{th} roots of 1 and -1 respectively, and are *frequently* written $(1)^{\frac{1}{n}}$ and $(-1)^{\frac{1}{n}}$: of the former one or

o, and of the latter *one* or *none*, being *arithmetical*, cording as n is *odd* or *even:* the sum of the roots in th cases being $= 0$.

Ex. To find the cube roots of 1 and -1.

(1) From $x = \cos \frac{2r\pi}{n} \pm \sqrt{-1} \sin \frac{2r\pi}{n}$, we have, putting 0, 1 for r,

$$x = 1: \text{ and } x = \cos \frac{2\pi}{3} \pm \sqrt{-1} \sin \frac{2\pi}{3}$$

$$= \cos 120^\circ \pm \sqrt{-1} \sin 120^\circ = -\cos 60^\circ \pm \sqrt{-1} \sin 60^\circ$$

$$= -\frac{1}{2} \pm \frac{\sqrt{-1}\sqrt{3}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.$$

(2) From $x = \cos \frac{(2r+1)\pi}{n} \pm \sqrt{-1} \sin \frac{(2r+1)\pi}{n}$,

e have, by putting 0, 1 for r;

$$x = \cos \frac{\pi}{3} \pm \sqrt{-1} \sin \frac{\pi}{3} = \cos 60^\circ \pm \sqrt{-1} \sin 60^\circ$$

$$= \frac{1}{2} \pm \frac{\sqrt{-1}\sqrt{3}}{2} = \frac{1 \pm \sqrt{-3}}{2}: \text{ and } x = -1.$$

Hence, the cube roots of ± 1, may be thus expressed:

$(1)^{\frac{1}{3}} = 1$, or $= \frac{1}{2}(-1 + \sqrt{-3})$, or $= \frac{1}{2}(-1 - \sqrt{-3})$:

$(-1)^{\frac{1}{3}} = -1$, or $= \frac{1}{2}(1 + \sqrt{-3})$, or $= \frac{1}{2}(1 - \sqrt{-3})$.

All these results are easily found by other methods: id the same process may be adopted whatever roots e required.

158. If $x^n \pm a^n = 0$, be the equation proposed, then y assuming $y = \frac{x}{a}$, or $x = ay$, we have

$$a^n y^n \pm a^n = 0, \text{ or } y^n \pm 1 = 0,$$

om which the values of y may be found as before, nd thence those of $x = ay$.

159. *To solve the equation* $x^{2n} - 2\cos a\, x^n + 1 = 0$.

By completing the square, &c. we have

$$^n = \cos a \pm \sqrt{-1} \sin a = \cos(2r\pi + a) \pm \sqrt{-1} \sin(2r\pi + a):$$

from which, by *Demoivre's* Theorem, we obtain

$$x = \cos\left(\frac{2r\pi + a}{n}\right) \pm \sqrt{-1}\,\sin\left(\frac{2r\pi + a}{n}\right):$$

where the values of x will be determined by assigning to r, the successive values 0, 1, 2, 3, &c, $n-1$.

Hence, proceeding as before, we shall find

$$x^{2n} - 2\cos a\, x^n + 1$$

$$= (x^2 - 2\cos\frac{a}{n}\,x + 1)\,(x^2 - 2\cos\frac{2\pi + a}{n}\,x + 1)$$

$$\&\text{c.}\ldots\ (x^2 - 2\cos\frac{2(n-1)\pi + a}{n}\,x + 1),$$

the number of quadratic factors being n: and thus by solving each of the resulting quadratic equations, we arrive at the complete solution of $x^{2n} - px^n + q = 0$, when p is not greater than 2, and $q = 1$.

160. Cor. Hence, making $x = 1$, we shall have

$$2 - 2\cos a = \left(2 - 2\cos\frac{a}{n}\right)\left(2 - 2\cos\frac{2\pi + a}{n}\right) \text{ \&c. to } n \text{ factors}$$

$$\text{or, } 2^2\sin^2\tfrac{1}{2}a = 2^{2n}\sin^2\frac{a}{2n}\,\sin^2\left(\frac{2\pi + a}{2n}\right) \text{ \&c.}:$$

and putting $2na$ for a, dividing by 2^2, &c., we have

$$\sin na = \pm 2^{n-1}\sin a\,\sin\left(a + \frac{\pi}{n}\right)\sin\left(a + \frac{2\pi}{n}\right) \text{ \&c. to } n \text{ factors}$$

Similar results may be obtained by making $x = -1$.

The Theorem of Article (159) is due to *M. Demoivre*; and by making a successively equal to 0 and π, we shall obtain the roots of the equations $x^n - 1 = 0$, and $x^n + 1 = 0$.

161. *To express the sine and cosine of an angle, by factors involving its circular measure.*

$$\text{Since, } \sin\theta = \theta\,(1 - \frac{\theta^2}{1.2.3} + \frac{\theta^4}{1.2.3.4.5} - \text{\&c.}):$$

the roots of the *equation* $\sin\theta = 0$, will be 0, $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$, &c.: whence, by the nature of equations, if s be a fixed magnitude, we have

$$\sin\theta = s\theta\,(\pi - \theta)\,(\pi + \theta)\,(2\pi - \theta)\,(2\pi + \theta) \text{ \&c.}$$

$$= s\theta\pi\left(1-\frac{\theta}{\pi}\right)\pi\left(1+\frac{\theta}{\pi}\right)2\pi\left(1-\frac{\theta}{2\pi}\right)2\pi\left(1+\frac{\theta}{2\pi}\right) \&c.$$

$$= s\pi^2(2\pi)^2(3\pi)^2\ \&c.\ \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right) \&c.:$$

but when $\theta=0$, we have seen that $\frac{\sin\theta}{\theta}=1$: and therefore we have $s\pi^2(2\pi)^2(3\pi)^2\ \&c. = 1$:

$$\text{whence, } \sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right) \&c.\ \textit{in infinitum.}$$

Again, $\cos\theta = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \&c.$: and the roots of the *equation* $\cos\theta = 0$, are $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, $\pm\frac{5\pi}{2}$, &c.:

$$\therefore\ \cos\theta = c\left(\frac{\pi}{2}-\theta\right)\left(\frac{\pi}{2}+\theta\right)\left(\frac{3\pi}{2}-\theta\right)\left(\frac{3\pi}{2}+\theta\right) \&c.$$

$$= c\frac{\pi}{2}\left(1-\frac{2\theta}{\pi}\right)\frac{\pi}{2}\left(1+\frac{2\theta}{\pi}\right)\frac{3\pi}{2}\left(1-\frac{2\theta}{3\pi}\right)\frac{3\pi}{2}\left(1+\frac{2\theta}{3\pi}\right) \&c.$$

$$= c\left(\frac{\pi}{2}\right)^2\left(\frac{3\pi}{2}\right)^2 \&c.\left(1-\frac{2^2\theta^2}{\pi^2}\right)\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right) \&c.:$$

but when $\theta = 0$, $\cos\theta = 1$, since c is fixed in value,

$$\text{and we have } c\left(\frac{\pi}{2}\right)^2\left(\frac{3\pi}{2}\right)^2 \&c. = 1:$$

$$\text{whence, } \cos\theta = \left(1-\frac{2^2\theta^2}{\pi^2}\right)\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right) \&c.\ \textit{in infinitum.}$$

162. Cor. 1. Let $\theta = \frac{1}{2}\pi$; then $\sin\theta = 1$, and we have $1 = \frac{\pi}{2}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right) \&c.$

$$= \frac{\pi}{2}\left(\frac{2^2-1}{2^2}\right)\left(\frac{4^2-1}{4^2}\right)\left(\frac{6^2-1}{6^2}\right) \&c.$$

$$= \frac{\pi}{2}\left\{\frac{(2-1)(2+1)}{2^2}\right\}\left\{\frac{(4-1)(4+1)}{4^2}\right\}\left\{\frac{(6-1)(6+1)}{6^2}\right\} \&c.$$

$$= \frac{\pi}{2}\left(\frac{1.3}{2^2}\right)\left(\frac{3.5}{4^2}\right)\left(\frac{5.7}{6^2}\right) \&c. = \frac{\pi}{2}\,\frac{3^2.5^2.7^2.9^2.\&c.}{2^2.4^2.6^2.8^2.\&c.}:$$

$$\text{whence, } \frac{\pi}{2} = \frac{2^2 . 4^2 . 6^2 . 8^2 . \text{\&c. } in\ inf.}{3^2 . 5^2 . 7^2 . 9^2 . \text{\&c. } in\ inf.};$$

which was first given by *Dr John Wallis*, an eminent Mathematician, some time *Savilian Professor of Geometry at Oxford*, who died in the year 1703.

This singular result might lead us to conclude that $\frac{1}{2}\pi$ is less than 1, because the factors in the numerator are less than those corresponding to them in the denominator; but it will easily be seen that in strict propriety,

$$\tfrac{1}{2}\pi = \frac{2^2 . 4^2 . 6^2 . 8^2 . \text{\&c. } in\ infinitum}{1^2 . 3^2 . 5^2 . 7^2 . \text{\&c. } in\ infinitum}:$$

and since no means yet discovered can prove that the latter quantity may be expressed in finite terms, π is always regarded as an *incommensurable* quantity.

The student may see this fact established beyond a doubt, in the Appendix to *Legendre's* Geometry, from the circumstance of π being expressible by an *irrational* continued fraction.

163. Cor. 2. In the expression above investigated, both the numerator and denominator are supposed to be continued *in infinitum:* if however, the number of factors taken be *very large* but *finite*, and be denoted by x, we shall have *approximately*,

$$1 = \frac{\pi}{2}\left(\frac{1.3}{2^2}\right)\left(\frac{3.5}{4^2}\right)\left(\frac{5.7}{6^2}\right)\text{\&c.}\left\{\frac{(2x-1)(2x+1)}{(2x)^2}\right\}$$

$$= \frac{\pi}{2}\left\{\frac{1^2 . 3^2 . 5^2 . \text{\&c. } (2x-1)^2 (2x+1)}{2^2 . 4^2 . 6^2 . \text{\&c. } (2x)^2}\right\}$$

$$= \frac{2x+1}{2}\,\pi\left\{\frac{1^2 . 3^2 . 5^2 . \text{\&c. } (2x-1)^2}{2^2 . 4^2 . 6^2 . \text{\&c. } (2x)^2}\right\}$$

$$= \pi x\left\{\frac{1^2 . 3^2 . 5^2 . \text{\&c. } (2x-1)^2}{2^2 . 4^2 . 6^2 . \text{\&c. } (2x)^2}\right\}, \text{ nearly}:$$

$$\text{and } \therefore \sqrt{\pi x} = \frac{2 . 4 . 6 . \text{\&c. } 2x}{1 . 3 . 5 . \text{\&c. } (2x-1)}, \text{ nearly.}$$

Some curious and interesting results, dependent upon the subjects of this chapter, will be found in the first *Appendix* annexed to the Work.

CHAPTER VII.

E CALCULATION OF LOGARITHMS, AND THE CONSTRUCTION OF MATHEMATICAL TABLES.

I. *The Calculation of Logarithms.*

164. From the exponential equation $y = a^x$, com- sing the definition of *Logarithms* as laid down in ticle (88), which is frequently written in the forms $_a y = x$, and $y = a^{\log_a y}$, we infer that when a is > 1, values of y increasing from 1 towards ∞ may be ob- ned by giving to x values increasing from 0 towards ∞: d when x is negative, or the equation becomes $y = \frac{1}{a^x}$, values of y decreasing from 1 towards 0 will be fur- hed by assigning to x values from 0 towards ∞: also logarithms of ∞ and 0 will both be infinite and have ferent algebraical signs. If $a < 1$, which may be sup- ed in *Theory*, but is never adopted in *Practice*, the soning may be conducted in a similar manner.

All negative bases will of course be excluded, because grand object of Logarithms is to facilitate *Arithme- l* Computations, not to mention that consecutive ues of y, within the limits above mentioned, could not obtained by assigning to x any continuous series of gnitudes whatever, inasmuch as some powers of a *ative* quantity would be *positive*, whilst others would *negative*.

From the view above given, it follows that the loga- ms of all numbers which are not *exact* powers or roots he base of the system will be *incommensurable* quan- es: but as absolute accuracy is *unattainable* in many the subjects to which Arithmetic is applied, and the roximations may be continued to the greatest degree icety, the errors may be made of such a magnitude as to be *appreciable*, in every *practical* or *scientific* appli- ion.

We are not *enabled* to express generally the value of *terms of a and y*, by any direct elementary method:

and recourse must therefore be had to that kind of *Analysis* which will give approximate results sufficiently accurate for the purposes to which Logarithms are intended to be applied: in pursuance of which, we shall first expand a^x in a *series* ascending by *positive integral* powers of x: and then point out the methods which are usually adopted to render it subservient to the actual calculation of the logarithm of any particular number proposed.

165. *To expand* a^x *in a series of ascending positive powers* of x.

Since
$$a^x = \{1 + (a-1)\}^x = 1 + x(a-1) + \frac{x(x-1)}{1.2}(a-1)^2 + \frac{x(x-1)(x-2)}{1.2.3}(a-1)^3 + \&c.,$$

by the Binomial Theorem:
$$= 1 + \{(a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.\}\,x + Bx^2 + Cx^3 + \&c.$$

where B, C, &c. comprise only $(a-1)$ and its powers: if the coefficient of x be assumed $= A$, we have

$$a^x = 1 + Ax + Bx^2 + Cx^3 + \&c. + Nx^{p-1} + Px^p + \&c.:$$
$$\therefore\ a^z = 1 + Az + Bz^2 + Cz^3 + \&c. + Nz^{p-1} + Pz^p + \&c.:$$

whence, by multiplication, we obtain

$$\begin{aligned} a^x \times a^z = 1 &+ Ax + Bx^2 + Cx^3 + \&c. + Px^p + \&c. \\ &+ Az + A^2xz + ABx^2z + \&c. + ANx^{p-1}z + \&c. \\ &+ \&c. \ldots\ldots\ldots\ldots \end{aligned}$$

but $a^x \times a^z = a^{x+z}$, by what precedes,

$$\begin{aligned} &= 1 + A(x+z) + B(x+z)^2 + C(x+z)^3 + \&c. + P(x+z)^p + \&c. \\ &= 1 + Ax + Bx^2 + Cx^3 + \&c. + Px^p + \&c. \\ &\quad + Az + 2Bxz + 3Cx^2z + \&c. + pPx^{p-1}z + \&c. \\ &\quad + \&c. \ldots\ldots\ldots\ldots \end{aligned}$$

now these two expansions of $a^x \times a^z$ must be *identical*, and therefore by equating coefficients in each, we have

$$2B = A^2, \text{ or } B = \frac{A^2}{1.2};$$

$$3C = AB = \frac{A^3}{1.2}, \text{ or } C = \frac{A^3}{1.2.3}: \&c. = \&c.$$

$$pP = AN, \text{ or } P = \frac{AN}{p}:$$

and from the last we infer that if the *law*, connecting two consecutive terms, hold good in *one* instance, it is *general:*

$$\therefore a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \&c. + \frac{A^px^p}{1.2.3\ldots p} + \&c.$$

This equality, called the *Exponential Theorem*, which is *arithmetically* correct when x is a positive whole number, or when the value of A is expressed in a *finite* number of terms, is here *assumed* to be general: and therefore in other cases, the value of $y = a^x$ will be obtained *approximately*, only when the series *converges*.

166. *From* $y = a^x$, *to express* x *in terms of* a *and* y.

$$\text{Since, } a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \&c.$$

where $A = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.$ *in inf.:*

$$\text{we shall have } y^z = 1 + Lz + \frac{L^2z^2}{1.2} + \frac{L^3z^3}{1.2.3} + \&c.$$

where $L = (y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \&c.$ *in inf.:* but from $y = a^x$, we have $y^z = a^{xz}$: and therefore

$$y^z = 1 + Axz + \frac{A^2x^2z^2}{1.2} + \frac{A^3x^3z^3}{1.2.3} + \&c.:$$

and the identity of these two series for y^z, gives throughout their extent, $L = Ax$, and $\therefore x = \frac{L}{A}$:

that is, x or $\log_a y = \dfrac{(y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \&c.}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.}$: which is the logarithm of y expressed in terms of y and the base a: but this result can evidently be of no practical use for the arithmetical calculation of x, unless both the numerator and denominator converge.

167. Cor. The expression for x, above found, will be *much simplified* by assuming the denominator $= 1$:

but it is manifest that this cannot be done without at the same time assigning a *particular* value to the base a: and from this circumstance arises the *peculiar* system adopted by *Lord Napier*, the *Inventor* of Logarithms, which will consequently possess great advantages in the facility with which the logarithms of numbers are calculated, though it is never used for the purposes of *Arithmetical* Computation.

This is therefore termed the *Napierian System* of Logarithms: and sometimes the *Hyperbolic System*, from a close analogy subsisting between such logarithms and the area of a rectangular *Hyperbola* referred to its asymptotes. It is regarded as the *Primitive* or *Natural* system, and is used in most parts of *Analysis:* and, with the exception of the *Common System*, which is universally applied to *Arithmetic*, it is the only system which we shall ever have occasion to consider.

168. *To calculate the numerical value of the Base of the Napierian System of Logarithms.*

Let e denote the value of a, which will render

$$A = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c. = 1:$$

then, if $A = 1$ in the series of Article (165), we have

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \&c.,$$

whereof the number of terms is indefinitely great: whence, making $x = 1$, we find

$$e = 1 + \frac{1}{1} + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \&c.:$$

now, deducing every term from the one *preceding* it,

$$1 + 1 = 2.$$

$$\frac{1}{1\,.\,2} = .5$$

$$\frac{1}{1\,.\,2\,.\,3} = .166666666 \;\&c.$$

$$\frac{1}{1\,.\,2\,.\,3\,.\,4} = .041666666 \;\&c.$$

$$\frac{1}{1\,.\,2\,.\,3\,.\,4\,.\,5} = .008333333 \;\&c.$$

$$\frac{1}{1.2.3.4.5.6} = .001388888 \text{ \&c.}$$

$$\frac{1}{1.2.3.4.5.6.7} = .000198412 \text{ \&c.}$$

$$\frac{1}{1.2.3.4.5.6.7.8} = .000024801 \text{ \&c.}$$

$$\frac{1}{1.2.3.4.5.6.7.8.9} = .000002755 \text{ \&c.}$$

$$\frac{1}{1.2.3.4.5.6.7.8.9.10} = .000000275 \text{ \&c. \&c.:}$$

hence, $e = 2.7182818$ nearly, the *Napierian Base:*

and $(e-1) - \frac{1}{2}(e-1)^2 + \frac{1}{3}(e-1)^3 -$ &c. *in inf.* $= 1$.

It will be shewn in the first Appendix that e is *inommensurable* in the decimal system of notation.

169. *If the logarithms of numbers in the Napierian ystem be given, the logarithms of the same numbers in ny other system may be found.*

Denoting the logarithms of y in the two systems hose bases are a and e, by x and x_1 respectively,

we have $y = a^x$ and $y = e^{x_1}$, so that $a^x = e^{x_1}$:

whence $a = e^{\frac{x_1}{x}}$, and $\therefore \frac{x_1}{x} = \log_e a$, by Article (164):

hat is, $x = \frac{1}{\log_e a} x_1$, or $\log_a y = \frac{\log_e y}{\log_e a}$: and the logaithms of numbers to the base a, will be derived from hose of the same numbers to the base e, by multiplying y $\frac{1}{\log_e a}$, which is the same for all values of y.

Conversely, *Napierian* logarithms may be obtained rom logarithms to the base a; for $a^{\frac{x}{x_1}} = e$, gives $\frac{x}{x_1} = \log_a e$, nd $\therefore x_1 = \frac{1}{\log_a e} x$, or, $\log_e y = \frac{\log_a y}{\log_a e}$: and indeed when he bases are a and b, the relation, $\log_a y \times \log_e a = \log_b y \times \log_e b$, enables us to pass immediately from either system to the other.

170. **Def.** The multiplier $1 \div \log_e a$, which connects the primitive system whose base is e, with any other system whose base is a, is termed the *Modulus* of the *latter* system: and $1 \div \log_e 10$ is therefore the modulus of the common System of Logarithms.

Also, from Article (166) it appears that

$$\log_e a = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c. = A.$$

Converging Series for the immediate calculation of Napierian Logarithms.

171. Since $A = 1$, we have, from Article (166),

$$\log_e y = (y-1) - \tfrac{1}{2}(y-1)^2 + \tfrac{1}{3}(y-1)^3 - \&c.:$$

whence, if $y = 1 + z$, and $\therefore y - 1 = z$, we obtain

$$\log_e(1+z) = z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \&c.:$$

which converges very slowly, unless z be very small.

172. From the expression for $\log_e y$ above given

if $y = \frac{1}{z}$, and $\therefore \log_e y = \log_e \frac{1}{z} = \log_e 1 - \log_e z = -\log_e z$:

$$\text{we have } -\log_e z = \left(\frac{1}{z} - 1\right) - \tfrac{1}{2}\left(\frac{1}{z} - 1\right)^2 + \tfrac{1}{3}\left(\frac{1}{z} - 1\right)^3 - \&c.$$

$$= \left(\frac{1-z}{z}\right) - \tfrac{1}{2}\left(\frac{1-z}{z}\right)^2 + \tfrac{1}{3}\left(\frac{1-z}{z}\right)^3 - \&c.:$$

$$\text{which gives } \log_e z = \frac{z-1}{z} + \tfrac{1}{2}\left(\frac{z-1}{z}\right)^2 + \tfrac{1}{3}\left(\frac{z-1}{z}\right)^3 + \&c.:$$

a series converging with tolerable rapidity, since each term is a *proper* fraction, and all the terms have the *same* algebraical sign.

Ex. If $z = 10$, we have $\frac{z-1}{z} = \frac{9}{10} = .9$:

$$\therefore \log_e 10 = .9 + \tfrac{1}{2}(.9)^2 + \tfrac{1}{3}(.9)^3 + \&c.:$$

but this requires a great number of terms to be taken, to give the value $= 2.302585076$ &c., hereafter found.

173. By the last article but one, we have

$$\log_e(1+z) = \quad z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \tfrac{1}{4}z^4 + \&c.:$$

$$\therefore \log_e(1-z) = -z - \tfrac{1}{2}z^2 - \tfrac{1}{3}z^3 - \tfrac{1}{4}z^4 - \&c.:$$

ce, by subtraction and Article (93), is obtained

$$\log_e \frac{1+z}{1-z} = 2\left\{z + \frac{1}{3}z^3 + \frac{1}{5}z^5 + \&c.\right\}:$$

herefore, if $\frac{1+z}{1-z} = u$, we have $z = \frac{u-1}{u+1}$, so that

$$\log_e u = 2\left\{\frac{u-1}{u+1} + \tfrac{1}{3}\left(\frac{u-1}{u+1}\right)^3 + \tfrac{1}{5}\left(\frac{u-1}{u+1}\right)^5 + \&c.\right\}:$$

since $\frac{u-1}{u+1}$ is a less quantity than $\frac{z-1}{z}$, if z be

e equal to u, it is evident that the series converges h more rapidly than the one found in the last article: it is well adapted to *small* prime numbers, because term is then considerably less than 1.

x. If $u = 2$, we have $\frac{u-1}{u+1} = \frac{1}{3}$: and by substitution,

$$\log_e 2 = 2\left\{\frac{1}{3} + \tfrac{1}{3}\left(\frac{1}{3}\right)^3 + \tfrac{1}{5}\left(\frac{1}{3}\right)^5 + \&c.\right\}:$$

its approximate value is calculated as follows:

$$\frac{1}{3} = .333333333 \text{ \&c.}$$

$$\frac{1}{3^3} = \frac{1}{3}\cdot\frac{1}{9} = .037037037 \text{ \&c.}$$

$$\frac{1}{3^5} = \frac{1}{3^3}\cdot\frac{1}{9} = .004115226 \text{ \&c.}$$

$$\frac{1}{3^7} = \frac{1}{3^5}\cdot\frac{1}{9} = .000457247 \text{ \&c.}$$

$$\frac{1}{3^9} = \frac{1}{3^7}\cdot\frac{1}{9} = .000050805 \text{ \&c.}$$

$$\frac{1}{3^{11}} = \frac{1}{3^9}\cdot\frac{1}{9} = .000005645 \text{ \&c.}$$

$$\frac{1}{3^{13}} = \frac{1}{3^{11}}\cdot\frac{1}{9} = .000000627 \text{ \&c.}$$

$$\frac{1}{3^{15}} = \frac{1}{3^{13}}\cdot\frac{1}{9} = .000000069 \text{ \&c. \&c.}:$$

whence, we have immediately

$$\frac{1}{3} = .333333333 \text{ &c.}$$

$$\frac{1}{3} \cdot \frac{1}{3^3} = .012345679 \text{ &c.}$$

$$\frac{1}{5} \cdot \frac{1}{3^5} = .000823045 \text{ &c.}$$

$$\frac{1}{7} \cdot \frac{1}{3^7} = .000065321 \text{ &c.}$$

$$\frac{1}{9} \cdot \frac{1}{3^9} = .000005645 \text{ &c.}$$

$$\frac{1}{11} \cdot \frac{1}{3^{11}} = .000000513 \text{ &c.}$$

$$\frac{1}{13} \cdot \frac{1}{3^{13}} = .000000048 \text{ &c.}$$

$$\frac{1}{15} \cdot \frac{1}{3^{15}} = .000000004 \text{ &c. &c.}:$$

$$\therefore \log_e 2 = 2\{.346573588 \text{ &c.}\} = .693147176 \text{ &c.}$$

Similarly, $\log_e 3 = 1.098612284$ &c.

Converging Series for deriving the Napierian L rithms of numbers, from those of inferior numbers.

174. Since $y = y - 1 + 1 = (y-1)\left(1 + \frac{1}{y-1}\right)$,

we have $\log_e y = \log_e (y-1) + \log_e \left(1 + \frac{1}{y-1}\right)$

$$\log_e (y-1) + \left\{\frac{1}{y-1} - \frac{1}{2}\,\frac{1}{(y-1)^2} + \frac{1}{3}\,\frac{1}{(y-1)^3} - \text{&c.}\right\}:$$

from which, if $\log_e (y-1)$ be given, $\log_e y$ may be fo also, the series converges very rapidly, but it is o tionable on account of the *alternation* of the algebı signs.

175. If $\log_e y$ be given, $\log_e (y+z)$ may be foun

For, $\log_e (y+z) = \log_e y\left(1 + \frac{z}{y}\right) = \log_e y + \log_e\big(1$

$$\log_e u = 2\left\{\frac{u-1}{u+1} + \frac{1}{3}\left(\frac{u-1}{u+1}\right)^3 + \frac{1}{5}\left(\frac{u-1}{u+1}\right)^5 + \&c.\right\},$$

if $u = 1 + \frac{z}{y}$, and $\therefore \frac{u-1}{u+1} = \frac{z}{2y+z}$, gives

$$\left(1 + \frac{z}{y}\right) = 2\left\{\frac{z}{2y+z} + \frac{1}{3}\left(\frac{z}{2y+z}\right)^3 + \frac{1}{5}\left(\frac{z}{2y+z}\right)^5 + \&c.\right\}:$$

$$\log_e(y+z) = \log_e y + 2\left\{\frac{z}{2y+z} + \frac{1}{3}\left(\frac{z}{2y+z}\right) + \&c.\right\}:$$

ch converges quickly, if z be small compared with y.

f $z = 1$, and y be a large number, we shall have

$$g_e(y+1) = \log_e y + 2\left\{\frac{1}{2y+1} + \frac{1}{3}\left(\frac{1}{2y+1}\right)^3 + \&c.\right\}:$$

erive the logarithm of a number from that of the inferior number, by means of a few terms.

Hence, the logarithm of 3 as given in Article (173), be found from that of 2 computed in it.

76. To deduce the logarithm of a number, from of *any* inferior number.

n $\log_e u = 2\left\{\frac{u-1}{u+1} + \frac{1}{3}\left(\frac{u-1}{u+1}\right)^3 + \frac{1}{5}\left(\frac{u-1}{u+1}\right)^5 + \&c.\right\}$,

let $u = \frac{p}{q}$, and $\therefore \frac{u-1}{u+1} = \frac{p-q}{p+q}$, and we shall have

$$p = \log_e q + 2\left\{\frac{p-q}{p+q} + \frac{1}{3}\left(\frac{p-q}{p+q}\right)^3 + \frac{1}{5}\left(\frac{p-q}{p+q}\right)^5 + \&c.\right\},$$

lly converging, if $p - q$ be small compared to $p + q$.

x. Let $p = 5$, $q = 4$: then $\log_e q = \log_e 4 = 2 \log_e 2$:

$$\therefore \log_e 5 = 2\log_e 2 + 2\left\{\frac{1}{9} + \frac{1}{3}\left(\frac{1}{9}\right)^3 + \frac{1}{5}\left(\frac{1}{9}\right)^5 + \&c.\right\}:$$

ow, $\frac{1}{9} = .111111111$ &c.

$$\frac{1}{9^3} = \frac{1}{9}(.012345679\ \&c.) = .001371742\ \&c.$$

$$\frac{1}{9^5} = \frac{1}{9}(.000152415 \text{ \&c.}) = .000016935 \text{ \&c.}$$

$$\frac{1}{9^7} = \frac{1}{9}(.000001881 \text{ \&c.}) = .000000209 \text{ \&c. \&c.}:$$

therefore, $\frac{1}{9} = .111111111$ &c.

$$\frac{1}{3} \cdot \frac{1}{9^3} = .000457247 \text{ \&c.}$$

$$\frac{1}{5} \cdot \frac{1}{9^5} = .000003387 \text{ \&c.}$$

$$\frac{1}{7} \cdot \frac{1}{9^7} = .000000029 \text{ \&c.}$$

$$\overline{\underline{.111571774 \text{ \&c.}}}$$

whence, $\log_e 5 = 2 \log_e 2 + 2(.111571774 \text{ \&c.})$

$= 1.386294352$ &c. $+ .223143548$ &c. $= 1.609437900$ &c.

177. Of three consecutive numbers, to obtain the logarithm of the greatest, from those of the two others:

Since, $\log_e(y+1) + \log_e(y-1) = \log_e(y^2-1)$

$$= \log_e y^2\left(1 - \frac{1}{y^2}\right) = 2\log_e y + \log_e\left(1 - \frac{1}{y^2}\right):$$

if we make $u = 1 - \frac{1}{y^2}$, and $\therefore \frac{u-1}{u+1} = -\frac{1}{2y^2-1}$,

we shall have from Article (173), $\log_e\left(1 - \frac{1}{y^2}\right)$

$$= -2\left\{\frac{1}{2y^2-1} + \frac{1}{3}\left(\frac{1}{2y^2-1}\right)^3 + \frac{1}{5}\left(\frac{1}{2y^2-1}\right)^5 + \text{\&c.}\right\}:$$

$$\therefore \log_e(y+1) = 2\log_e y - \log_e(y-1)$$

$$-2\left\{\frac{1}{2y^2-1} \quad \frac{1}{3}\left(\frac{1}{2y^2-1}\right)^3 + \frac{1}{5}\left(\frac{1}{2y^2-1}\right)^5 + \text{\&c.}\right\}:$$

and the last part of the expression converges very rapidly, when y is a large number.

178. Cor. If a, b, c denote the three numbers: then $a = b - 1$, $c = b + 1$, and $\therefore b^2 = ac + 1$:

$$\therefore 2 \log_e b = \log_e (ac + 1) = \log_e ac \left(1 + \frac{1}{ac}\right)$$

$$= \log_e a + \log_e c + \log_e \left(1 + \frac{1}{ac}\right):$$

but if $u = 1 + \dfrac{1}{ac}$, we have $\dfrac{u-1}{u+1} = \dfrac{1}{2ac+1}$:

$$\therefore \log_e \left(1 + \frac{1}{ac}\right)$$

$$= 2 \left\{\frac{1}{2ac+1} + \frac{1}{3}\left(\frac{1}{2ac+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2ac+1}\right)^5 + \&c.\right\}:$$

and $\log_e b = \frac{1}{2} \log_e a + \frac{1}{2} \log_e c$

$$+ \left\{\frac{1}{2ac+1} + \frac{1}{3}\left(\frac{1}{2ac+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2ac+1}\right)^5 + \&c.\right\}:$$

formula well suited for the calculation of the logarithms of *large* prime numbers, because if b be a prime umber, $a = b - 1$ and $c = b + 1$ are necessarily composite umbers, and therefore their logarithms may be found rom those of their factors, by Article (93).

This series converges so rapidly, that a very small umber of its terms is generally sufficient: thus, by aking only *three* terms, the logarithm of 5 will be obained correctly to *twelve* places of decimals.

If $b = 59$, and $\therefore a = 58$ and $c = 60$; we have

$$ac = 3480, \text{ and } \frac{1}{2ac+1} = \frac{1}{6961}:$$

nd in this instance, the *second* term of the series will not ffect the result, because it will have no significant figure n the first *twelve* places.

From this article is easily obtained the formula,

$$\text{og}_e\, c = 2 \log_e b - \log_e a - 2\left\{\frac{1}{2b^2-1} + \frac{1}{3}\left(\frac{1}{2b^2-1}\right)^3 + \&c.\right\}:$$

nd by judicious substitutions, a variety of others may be deduced, wherein the series converge with extraordinary rapidity, though they can never be of much use, because tables constructed to every degree of nicety requisite for *practical* purposes are already in our hands: thus,

$$\log_e (y + 2h) = 2 \log_e (y + h) - \log_e y$$
$$- \left\{ \frac{h^2}{(y+h)^2} + \tfrac{1}{2} \frac{h^4}{(y+h)^4} + \tfrac{1}{3} \frac{h^6}{(y+h)^6} + \&c. \right\}:$$

$$\log_e (y + 2h) = 2 \log_e (y + h) - 2 \log_e (y - h) + \log_e (y - 2h)$$
$$+ 2 \left\{ \frac{2h^3}{y^3 - 3h^2 y} + \tfrac{1}{3} \frac{8h^9}{(y^3 - 3h^2 y)^3} + \&c. \right\}.$$

179. By these and similar formulæ, it is now evident that the *Napierian* logarithms of all numbers whatever might be computed: but the properties, established in Article (93), render it necessary to apply them only to such numbers as are not resolvable into factors, because the logarithm of every *composite* number is equal to the sum of the logarithms of its *prime* factors.

Thus, $\log_e 2$, $\log_e 3$, and $\log_e 5$ have been found: and

$$\log_e 4 = \log_e 2^2 = 2 \log_e 2:$$
$$\log_e 6 = \log_e (2 \times 3) = \log_e 2 + \log_e 3:$$
$$\log_e 8 = \log_e 2^3 = 3 \log_e 2:$$
$$\log_e 9 = \log_e 3^2 = 2 \log_e 3:$$
$$\log_e 10 = \log_e (2 \times 5) = \log_e 2 + \log_e 5:$$
$$\log_e 12 = \log_e (3 \times 4) = \log_e 3 + \log_e 4: \ \&c.:$$

but to determine the logarithms of the prime numbers 7, 11, 13, &c., one or other of the preceding formulæ will be required: and the same for larger numbers.

Thus, if the proposed number be 1767, we have

$$\log_e 1767 = \log_e (6^2 . 7^2 + 3) = \log_e 6^2 . 7^2 \left(1 + \frac{3}{6^2 . 7^2}\right)$$
$$= 2 \log_e 6 + 2 \log_e 7 + \log_e \left(1 + \frac{1}{588}\right):$$

which may be computed by the formula of Article (171), to a great degree of accuracy.

180. *To calculate the numerical value of the Modulus of the Common System of Logarithms.*

In Article (170), we have seen that the Modulus $= 1 \div \log_e 10$, so that from Articles (173) and (176)

$$\begin{aligned} \text{we have, } \log_e 10 &= \log_e 2 + \log_e 5 \\ &= .693147176 \text{ \&c.} \\ &\quad + 1.609437900 \text{ \&c.} \\ &\quad\; 2.302585076 \text{ \&c.} \end{aligned}$$

$$\therefore \text{ the modulus} = \frac{1}{2.302585076 \text{ \&c.}} = .43429448 \text{ \&c.}$$

Hence, by Article (169), we have the equality

$$\log_{10} y = .43429448 \text{ \&c.} \times \log_e y = M \log_e y,$$

for deriving the *common* logarithm of a number from its *Napierian* logarithm, M being $= .43429448$ &c.

Ex. $\text{Log}_e 2 = .693147176$ &c.: $\log_e 5 = 1.609437900$ &c.:

$$\begin{aligned} \therefore \log_{10} 2 &= .43429448 \text{ \&c.} \times .693147176 \text{ \&c.} \\ &= .3010300 \text{ \&c.}: \\ \log_{10} 5 &= .43429448 \text{ \&c.} \times 1.609437900 \text{ \&c.} \\ &= .6989700 \text{ \&c.} \end{aligned}$$

181. The *Rules* for using the Tables, enunciated in full, may be seen in the introduction to *Babbage's* tables or in the chapters prefatory to those of *Dr Hutton*, in which the subject is treated at considerable length: and it now remains only to investigate the reason of the assumptions made in Articles (100) and (101).

182. *To investigate a formula for the approximate logarithms of numbers, consisting of more places of figures than those in the tables.*

Let y be the number in the tables, and $y' = y + z$, where y consists of *five* figures and z is a *decimal:* then

$$\log y' = \log (y + z) = \log y + \log \left(1 + \frac{z}{y}\right):$$

$$\textit{but } \log \left(1 + \frac{z}{y}\right) = M \left\{\frac{z}{y} - \frac{1}{2}\left(\frac{z}{y}\right)^2 + \frac{1}{3}\left(\frac{z}{y}\right)^3 - \text{\&c.}\right\},$$

where M denotes the modulus of the common system of Logarithms: and y being at least a number of *five* places of figures in the ordinary tables, the *greatest* value which $\frac{z}{y}$ can have, will evidently be $\frac{1}{10000} = .0001$:

$$\text{whence, } \log y' = \log(y+z) = \log y + M\frac{z}{y}, \text{ nearly:}$$

$$\text{also, } \log(y+1) - \log y = M\frac{1}{y}, \text{ nearly, if } z = 1:$$

$$\therefore M = \{\log(y+1) - \log y\}y, \text{ nearly}:$$

$$\text{and } \log y' = \log y + \{\log(y+1) - \log y\}\, z, \text{ nearly}:$$

hence, $\log y'$ is found by means of $\log(y+1)$ and $\log y$, between which it lies, with sufficient exactness for most practical purposes.

If y' be a whole number, and z comprise 1, 2, &c. digits, we have only to increase the characteristic by 1, 2, &c.

If $z = \frac{h}{10} + \frac{k}{100} + \&\text{c.}$, or the *additional* digits h, k, &c. occupy the places of *tenths*, *hundredths*, &c.,

$$\log y' = \log y + \{\log(y+1) - \log y\}\left\{\frac{h}{10} + \frac{k}{100} + \&\text{c.}\right\}.$$

183. This last result is the foundation of the small tables, before called *Proportional parts;* and they are constructed on the principle, which is implied in

$$\frac{\log y' - \log y}{\log(y+1) - \log y} = \frac{h}{10} + \frac{k}{100} + \&\text{c.}:$$

that the increments of the *Logarithms* of numbers are *proportional* to the increments of the *Numbers* themselves; but this can hold good only, when the numbers consist of several places of figures, as above shewn. See Articles (100) and (104).

184. *To find the number corresponding to a logarithm, not found exactly in the tables.*

Let x be the proposed logarithm, y the corresponding number: x_0 and x_1 the next less and greater tabular logarithms, corresponding to the numbers y_0 and y_0+1:

nd suppose $y = y_0 + z$, where z is *a fractional* quantity *nnexed* to the right of y_0: then

$$= \log (y_0 + z) = \log y_0 + \log \left(1 + \frac{z}{y_0}\right) = x_0 + M \frac{z}{y_0}, \text{ nearly}:$$

$$\text{also, } x_1 = \log (y_0 + 1) = x_0 + M \frac{1}{y_0}, \text{ nearly, if } z = 1:$$

$$\text{whence, } z = \frac{x - x_0}{x_1 - x_0}: \text{ and } \therefore\ y = y_0 + \frac{x - x_0}{x_1 - x_0}, \text{ nearly,}$$

hich is consequently found. See Articles (101) and 104).

II. *The Construction of Mathematical Tables.*

185. It has been seen in Article (67), by what means e numerical values of the angular functions are directly etermined: but it is not necessary to compute the *sines* nd *cosines* of angles at all, in order to obtain their *logaithms,* as will now be made to appear.

186. *To find the logarithmic sine and cosine of an ngle, independently of its natural sine and cosine.*

In Article (161), it has been proved that

$$\sin a = a\left(1 - \frac{a^2}{\pi^2}\right)\left(1 - \frac{a^2}{2^2\pi^2}\right)\left(1 - \frac{a^2}{3^2\pi^2}\right) \&c.:$$

$$\text{whence, if } a \text{ be assumed} = \frac{m}{n}\frac{\pi}{2}, \text{ we shall have}$$

$$\log \sin\left(\frac{m}{n}\frac{\pi}{2}\right) = \log\left(\frac{m}{n}\frac{\pi}{2}\right) + \log\left(1 - \frac{m^2}{2^2 n^2}\right)$$
$$+ \log\left(1 - \frac{m^2}{4^2 n^2}\right) + \&c.:$$

nd each of the terms of the latter member after the irst, being expanded by the formula of Article (171),

$$\log \sin\left(\frac{m}{n}\frac{\pi}{2}\right)$$

will be obtained by means of converging series.

Also, *from* $\cos a = \left(1 - \dfrac{2^2 a^2}{\pi^2}\right)\left(1 - \dfrac{2^2 a^2}{3^2 \pi^2}\right)$ &c.

$$\log \cos\left(\frac{m}{n}\,\frac{\pi}{2}\right) = \log\left(1 - \frac{m^2}{n^2}\right) + \log\left(1 - \frac{m^2}{3^2 n^2}\right) + \&c.$$

In both these expressions, m and n may be assumed at pleasure, provided $m < n$: and thus, the *Tabular* logarithmic sines and cosines of all angles less than 90°, will be obtained by the addition of 10 to each.

Because $\frac{m}{n}$ is not necessarily a *small* fraction, the logarithm of the first binomial, in each of these equalities, is not expanded, inasmuch as the corresponding series may not converge with any great rapidity.

See *Woodhouse's* Trigonometry; also, *Callet's* Logarithms, and the *Encyclopædia Metropolitana.*

187. Cor. The logarithmic sines and cosines being determined, the other logarithmic functions are immediately deduced if necessary: thus,

$$L \tan a = 10 + L \sin a - L \cos a:$$
$$L \sec a = 20 - L \cos a: \ \&c.$$

188. The logarithms of the sines and cosines thus computed, may be *verified* by such formulæ as,

$$\sin n\theta = 2^{n-1} \sin\theta \sin\left(\frac{\pi}{n} + \theta\right) \sin\left(\frac{2\pi}{n} + \theta\right) \&c.:$$

where the number of trigonometrical factors in the latter member is n, as established in Article (160).

189. The object of the remaining articles will be to ascertain, *to what extent,* the Principle of *Proportional Parts* is applicable to the determination of the angular functions, and also of their logarithms, when they contain *seconds*, &c., and the converse.

190. *To find the increment of the sine of an angle, corresponding to a small increment of the angle itself.*

Let h denote a small increment of the angle A: then, the increment of the sine of A expressed by $\Delta \sin A$

$$= \sin(A + h) - \sin A$$
$$= \sin A \cos h + \cos A \sin h - \sin A$$
$$= \cos A \sin h - \sin A\,(1 - \cos h)$$

$$= \cos A \sin h - \sin A\,(1 - \sqrt{1 - \sin^2 h})$$
$$= \cos A \sin h - \sin A\,(\tfrac{1}{2}\sin^2 h + \tfrac{1}{8}\sin^4 h + \&c.)$$
$$= \cos A \sin h - \tfrac{1}{2}\sin A \sin^2 h\,(1 + \tfrac{1}{4}\sin^2 h + \&c.)$$
$$= \cos A \sin h\,\{1 - \tfrac{1}{2}\tan A \sin h\,(1 + \tfrac{1}{4}\sin^2 h + \&c.)\}:$$

hich is a general expression whatever the magnitude of may be: now, if h and therefore $\sin h$ be very small, nd $\tan A$ be not very large, we shall have

$$\Delta \sin A = \cos A \sin h, \text{ nearly}:$$

nd when h is given, it follows that in this case

$$\Delta \sin A \textit{ varies } \text{as } \cos A, \text{ nearly}:$$

r, the *increment* of the *sine* is very nearly proportional the *cosine*, when the angle is not nearly equal to 90°.

If however $\tan A$ be very *large*, which is the case hen A is *nearly* equal to 90°, $\frac{1}{2}\tan A \sin h + \&c.$ will ot be *small* compared to 1, and therefore the expression $\sin A \propto \cos A$, will no longer be *approximately* true.

191. Precisely in the same manner: or, by the substitution of $90^\circ - A$ in the place of A, it is shewn that

$$\cos A = -\sin A \sin h\,\{1 + \tfrac{1}{2}\cot A \sin h\,(1 + \tfrac{1}{4}\sin^2 h + \&c.)\}:$$

nd if $\cot A$ be not very large, we shall have

$$\Delta \cos A = -\sin A \sin h, \text{ nearly}:$$

r, the *decrement* of the *cosine* is nearly proportional to he *sine*, when the angle is not nearly = to 0°, or 180°.

When A is nearly = 0°, or 180°, we have

$$\Delta \cos A = -\sin A \sin h\,(1 + \tfrac{1}{2}\cot A \sin h), \text{ nearly}:$$

nd the proportionality just mentioned is no longer true.

192. *To find the increment of the tangent of an angle, orresponding to a small increment of the angle itself.*

Here, $\Delta \tan A = \tan(A + h) - \tan A$

$$= \frac{\sin(A+h)}{\cos(A+h)} - \frac{\sin A}{\cos A}$$
$$= \frac{\sin(A+h)\cos A - \cos(A+h)\sin A}{\cos A \cos(A+h)}$$
$$= \frac{\sin h}{\cos A\,(\cos A \cos h - \sin A \sin h)}$$

$$= \frac{\sin h}{\cos^2 A \cos h\,(1 - \tan A \tan h)}$$

$$= \sec^2 A \tan h \left(\frac{1}{1 - \tan A \tan h}\right).$$

If A be not nearly $= 90^\circ$, we have, as before,

$$\Delta \tan A = \sec^2 A \tan h \propto \sec^2 A, \text{ nearly}:$$

but this will not hold good, whenever $\tan A$ is a very *large* magnitude, or A is *nearly* $= 90^\circ$.

Similarly, $\Delta \cot A = -\operatorname{cosec}^2 A \tan h$, nearly, $\propto -\operatorname{cosec}^2 A$, nearly, when $\cot A$ is not very *large*, or when A is not *nearly* $= 0^\circ$, or 180°.

The same method may be applied to determine

$$\Delta \sec A,\ \Delta \operatorname{cosec} A,\ \&c.$$

193. Cor. When A does not possess the extreme values above excepted, it is evident that the variation in $\sin A$ will be *greater than*, *equal to*, or *less than* the variation in $\cos A$, according as A is *less than*, *equal to*, or *greater than* 45°.

These two propositions tell us in the language of *Algebra*, what has been seen *arithmetically* in Exs. 3 and 4, of Article (104).

194. *The small changes in the angular functions, are nearly proportional to the increment of the angle, except in the cases above mentioned.*

For, from Article (190), we have approximately,

$$\sin (A + h) - \sin A = \cos A \sin h:$$

$$\sin (A + k) - \sin A = \cos A \sin k:$$

whence, $\dfrac{\sin (A + h) - \sin A}{\sin (A + k) - \sin A} = \dfrac{\sin h}{\sin k} = \dfrac{h}{k}$, nearly.

If $k = 1' = 60''$, and h be less than $60''$, we have

$$\frac{\sin (A + h) - \sin A}{\sin (A + 1') - \sin A} = \frac{h}{60}:$$

from which, if $\sin A$ and $\sin (A + 1')$ be given, $\sin (A + h)$ may be found when h is given, and h may be found when $\sin (A + h)$ is given.

The same may be proved of the other functions; and appears that in the complemental functions, *subtraction* ust be used, whenever *addition* takes place in the rest.

195. Cor. 1. From this proposition it is manifest hat the Principle of *Proportional parts* may, with the xceptions above stated, be applied to determine the unctions of angles containing *seconds*, &c., from those of he next inferior angles calculated for minutes, and *vice ersâ:* though in some cases it would require them to be omputed to more than *seven* places of decimals. See rticle (105).

196. Cor. 2. For angles less than 45°, $\tan A$ is less an $\cot A$, and $\therefore\ \Delta \sin A$ is more nearly $= \cos A \sin h$, an $\Delta \cos A$ is to $-\sin A \sin h$: or, the proportional parts ill be more accurately ascertained for the sine, than e cosine: and the contrary for angles greater than 45°.

Thus, in finding $\sin \frac{1}{2} A$, where A is small, whatever rror may be expected in applying the formula,

$$\sin \tfrac{1}{2} A = \sqrt{\frac{1-\cos A}{2}}:$$

smaller one may be looked for in using

$$\sin \tfrac{1}{2} A = \tfrac{1}{2}\sqrt{1+\sin A} - \tfrac{1}{2}\sqrt{1-\sin A}.$$

197. *The small changes in the logarithmic functions f an angle, are nearly proportional to the increment of the ngle, except in the cases above mentioned.*

$$\text{For, } \frac{\sin(A+h)}{\sin A} = \frac{\sin A \cos h + \cos A \sin h}{\sin A}$$

$$= \cos h + \cot A \sin h = 1 + \cot A \sin h, \text{ nearly:}$$

$$\therefore \log \sin(A+h) - \log \sin A = \log(1 + \cot A \sin h)$$

$$= M \cot A \sin h, \text{ nearly, by Article (171):}$$

$$\text{whence, } \frac{\log \sin(A+h) - \log \sin A}{\log \sin(A+k) - \log \sin A} = \frac{\sin h}{\sin k} = \frac{h}{k}, \text{ nearly.}$$

If $k = 1' = 60''$, as before, and h be less than $60''$,

$$\frac{L \sin(A+h) - L \sin A}{L \sin(A+1') - L \sin A} = \frac{h}{60},$$

from which, inferences similar to those of Article (194), may be drawn.

A similar proof may be adopted for each of the other logarithmic functions, and the observations of Articles (195) and (196) may be repeated here.

198. In consequence of the tables containing only *approximate* values of the Logarithmic Functions to seven places of decimals, it will sometimes happen that the same function may lead to several angles differing by 1″; thus in the last formula, if $A = 89^\circ\, 7'$, we find

$$L \sin (A + 1') - L \sin A = .0000019,$$

so that the value of

$$L \sin (A + h) = L \sin A + \frac{h}{60} (.0000019),$$

will not differ from that of $L \sin A$, when the values of h are 1, 2, and 3: and recourse must be had to expedients to avoid this circumstance whenever great accuracy is required.

We can easily find the limits within which this may be expected to take place: for since

$$\triangle L \sin A = L \sin (A + h) - L \sin A = M \cot A \sin h;$$

at the *limit*, when $\sin h = \sin 1''$, we have $\cot A = \dfrac{.0000001}{M \sin 1''}$, which, by putting $M = .43429448$ &c., and

$$\sin 1'' = .00000484813 \text{ \&c.},$$

gives an angle very nearly equal to $87^\circ\, 19'$, beyond which it will not be safe to go without tables calculated to a greater extent of decimals.

199. For certain practical purposes it is sometimes requisite to determine the *numbers* of seconds in very small angles or arcs from the logarithms of their sines or tangents; and here the *logarithmic* functions may be subject to the inconvenience just mentioned, but the numbers of seconds corresponding to them will be obtained to a great degree of nicety as follows.

(1) Let the logarithmic sine of the angle or arc be given.

Since, $\sin\alpha = \alpha - \frac{\alpha^3}{1.2.3} = \alpha\left(1 - \frac{\alpha^2}{1.2}\right)^{\frac{1}{3}} = \alpha(\cos\alpha)^{\frac{1}{3}}$, nearly, we have $L\sin\alpha = 10 + \log\alpha - \frac{1}{3}(10 - L\cos\alpha)$, nearly: now, if n be the number of *seconds* contained in α,

$$\frac{\alpha}{\pi} = \frac{n}{180 \times 60 \times 60}, \text{ or } \alpha = \frac{n\pi}{180 \times 60 \times 60}:$$

$$\therefore \log\alpha = \log n + \log 3.14159 \text{ \&c.} - \log(180 \times 60^2)$$
$$= \log n - 5.3144251 \text{ \&c. by the tables:}$$

whence, $\log n = \log\alpha + 5.3144251$ &c.

$= L\sin\alpha - 4.6855749$ &c. $+ \frac{1}{3}$ (Arith. Comp. of $L\cos\alpha$).

Because $n = \frac{\alpha}{\sin 1''}$, nearly, by Article (129),

we have $\sin\alpha = n\sin 1''(\cos\alpha)^{\frac{1}{3}}$, nearly,

which gives $\log n = L\sin\alpha + \frac{1}{3}(10 - L\cos\alpha) - L\sin 1''$: and thus it appears that $L\sin 1'' = 4.6855749$ &c., as may easily be verified.

(2) Let the logarithmic *tangent* be given.

Since, from (1), $\sin\alpha = \alpha(\cos\alpha)^{\frac{1}{3}}$, we have $\tan\alpha = \frac{\alpha}{(\cos\alpha)^{\frac{2}{3}}}$,

whence, $L\tan\alpha = 10 + \log\alpha + \frac{2}{3}(10 - L\cos\alpha)$:

$\therefore \log n = L\tan\alpha - 4.6855749$ &c. $- \frac{2}{3}(10 - L\cos\alpha)$,

by proceeding as before; and we have likewise

$$\log n = L\tan\alpha - \tfrac{2}{3}(10 - L\cos\alpha) - L\sin 1''.$$

200. The former of these results was first given by *Dr. Maskelyne,* Astronomer Royal at *Greenwich*, in his Introduction to *Taylor's* Logarithms: but *M. Delambre* constructed a table for every *second* up to a *degree* for $\log\left(\frac{\sin\alpha}{\alpha}\right)$, which will answer the same purpose.

Thus, since $\frac{\sin\alpha}{\alpha} = 1 - \frac{1}{6}\alpha^2 + \frac{1}{120}\alpha^4$, nearly,

we shall have from Article (171), very nearly,

$$\log\left(\frac{\sin\alpha}{\alpha}\right) = -M\left\{\left(\frac{1}{6}\alpha^2 - \frac{1}{120}\alpha^4\right) + \tfrac{1}{2}\left(\frac{1}{6}\alpha^2 - \frac{1}{120}\alpha^4\right)^2\right\},$$

from which, if a be the number corresponding to the number n of seconds contained in α, we obtain

$$\log\left(\frac{\sin n''}{n''}\right) = a, \text{ and } \therefore \log n = \log \sin n'' - a,$$

$$\text{or, } \log n = L \sin n'' - (a + 10).$$

Because $\log\left(\frac{\sin \alpha}{\alpha}\right) = \log\left(\frac{\sin n''}{n \sin 1''}\right)$, nearly, we have

$$\log n = L \sin n'' - \left\{\log \frac{\sin \alpha}{\alpha} + L \sin 1''\right\},$$

which will afford extreme facility for computation, if the quantity between the brackets be tabulated for every second.

See *Callet's* Logarithms before referred to: and in *Dr. Hutton's* Mathematical Tables, the Student will find what is called a table of *Logistic* Logarithms, where the tabulated numbers are the differences between the common logarithm of $3600'' = 1°$, and those of any number of seconds up to $80'$ or $4800''$; and this is applicable to the purpose contemplated in the present Article.

SPHERICAL TRIGONOMETRY.

CHAPTER I.

DEFINITIONS AND PRELIMINARY PROPOSITIONS.

ARTICLE I. DEFINITION.

Spherical Trigonometry, in its strict meaning, is hat part of the Science which treats of the relations between the sides and angles of triangles, formed by the ntersections of three planes with the surface of a *Globe* or *Sphere*, in contradistinction to *Plane Trigonometry*, which was originally confined to plane triangles.

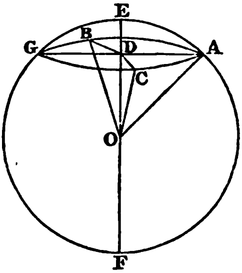
In reality, however, the object of Spherical Trigonometry is to determine the relations between three ectilineal *Angles*, forming what is called a *Solid Angle*, and the *Inclinations* of their *Planes* to each other; the atter being termed *Dihedral Angles*, to distinguish them from those contained by the *Edges* of the plane *Faces*, which constitute the solid angle.

The Student will acquire a correct idea of what is mplied in this definition, by dividing a circular *sector*, drawn on *pasteboard*, into *three* sectors such that the angles of any two of them may together be *greater* than that of the third, cutting the pasteboard half through, and turning up the *extreme* sectors so as to form a *space*, or *cavity*, analogous to the *sector* of a *sphere*.

2. Def. If with the angular point as a centre, the surface of a sphere be described cutting the *three* planes by which the solid angle is formed; it is evident that the portion of the surface of the sphere, included between their intersections, will have the same relation to the angular space which they contain, as the arc of a circle has to the angle which it subtends at the centre.

The *Lines*, which enclose this portion of the surface, are called the *Sides* of the *Spherical* Triangle which they form, and the *Inclinations* of their planes to each other, are termed its *Angles*.

3. *Every Section of the Surface of a Sphere, made by a plane cutting it, is the circumference of a Circle.*



Let O be the centre of the sphere, $ABGC$ the section made by a plane passing through it: draw OD perpendicular to this plane, and produce it both ways to meet the surface in E and F: join AD, BD, CD, and draw the radii of the sphere OA, OB, OC; then by *Euclid* XI. Def. 3. the angles ODA, ODB, ODC are right angles:

$$\begin{aligned} \therefore\ DA^2 &= OA^2 - OD^2 \\ &= OB^2 - OD^2 = DB^2 \\ &= OC^2 - OD^2 = DC^2, \text{ \&c.}: \end{aligned}$$

whence $DA = DB = DC =$ &c., and therefore the section $ABGC$ is a circle whose centre is D, and whose radius

$$= DA = DB = DC = \text{\&c.}$$

Also, if E be joined with B, C, &c., we shall have

$$EB^2 = ED^2 + DB^2 = ED^2 + DC^2 = EC^2 = \text{\&c.}:$$

and therefore the distances of E from every point in the circumference of the section are equal.

4. Cor. If the *distance* of the cutting plane from the *centre* of the sphere be k, and the radius of the sphere be r, then the radius of the section $= \sqrt{r^2 - k^2}$: and thus, the *magnitude* of the circle is found, if its *position* be given: and *vice versâ*.

5. Def. When the cutting plane passes through *the centre* of the sphere, we have $k = 0$: or the radius of

he *section* is equal to the radius of the *sphere,* and the ircle is called a *Great Circle* of the sphere: in all other ases, it is termed a *Small Circle.*

6. Def. The points *E, F*, in which the line *OD*, *erpendicular* to the plane of the section, meets the urface of the sphere, are called the *Poles* of the circle *BGC*.

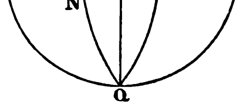
In the diagram, *E* is termed the *near,* and *F* the *emote* pole of the *small* circle *ABGC*: and for a *great* ircle, whose plane passes through *O,* at right angles o *EF,* the two poles are *equally* distant from it.

7. Cor. Hence, the poles of a circle are those points n the surface of the sphere, from which all chords and rcs, to its circumference, are respectively equal.

Also, since *EF* is perpendicular to the plane *ABGC,* very plane passing through *EF* cuts the plane *ABGC* at *ight angles:* that is, *any* circle of the sphere is at right ngles to *every* great circle passing through its poles.

8. *The arc of a great circle of the sphere, intercepted etween the circumference of a great circle and either of ts poles, is a quadrant.*

Let *AMBN* be a great circle, *O* its centre, *P* its pole:



PMQN a great circle passing through *P*: draw the dianeters *POQ, MON*: then, since *PO* is at right angles o the plane *AMBN,* the angles *POA, POM, PON,* are ight angles: whence it follows that

$$PM = PN = PA = \text{a quadrant.}$$

imilarly, $QM = QN = QA =$ a quadrant.

9. Cor. 1. Hence, if from any point P, two great circles be drawn to meet another great circle $AMBN$, in A and N, such that $PA = PN =$ a quadrant: then will P be the pole of the circle $AMBN$.

For, if any point R be taken in AN, and OR be drawn: then $\angle POR = \angle POA =$ a right angle,

and $PR = PA =$ a quadrant:

$\therefore$ by Article (7), P is the pole of the circle $AMBN$.

The circles $AMBN$ and $PMQN$ are termed *secondaries* to each other.

10. Cor. 2. Conversely, if two great circles PAQ, PNQ, be drawn at right angles to the circle $AMBN$, they will intersect each other in its poles.

For, since the planes $PAQB$, $PNQM$, are at right angles to the plane $AMBN$, their common section POQ is at right angles to the same plane: that is,

$\angle PON = \angle POA =$ a right angle,

and therefore $PN = PA =$ a quadrant; or P is the pole of the circle $AMBN$.

11. Cor. 3. From what has been said, it is evident that the arc of a great circle, intercepted between the poles of any two circles, measures their *inclination* to each other: and also that two great circles *bisect* each other, their common section being a diameter of each.

12. Def. The arcs on the surface of a sphere which form the *Sides* of a spherical triangle, are always understood to be portions of great circles, unless the contrary be expressed, and they are the measures of the plane angles which they subtend at its centre.

The sides are usually denoted by the letters a, b, c.

13. Def. The *Angles* of a spherical triangle are the inclinations of their planes to each other, and will therefore be equal to the rectilineal angles determined by drawing from any point in the common sections of the planes, a straight line in each plane at right angles to *these sections*: or to those formed by the tangents of the arcs by which they are contained.

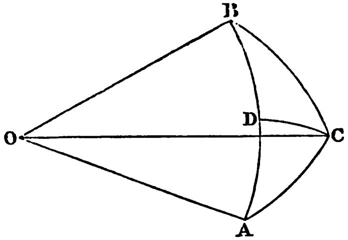
The letters A, B, C express the angles opposite to the sides a, b, c respectively.

14. Since the arc AN measures the *plane* angle AON, which is equal to the *spherical* angle APN, it follows that any angle of a spherical triangle may be measured by the corresponding arc of the great circle of which the *angular* point is the pole: also, the same angle may be measured by means of the arc of a *small circle* having the same pole: for, in the *circular measure,*

$$\angle\, AON = \angle\, aon = \frac{an}{ao} = \frac{an}{\sin Pa}.$$

The sides and angles of spherical quadrilaterals and polygons are understood in the same sense. If the radius of the sphere be *unity*, the functions of angles and arcs coincide, as appears from Article (33), *Pl. Tr.*

15. *Any two sides of a spherical triangle are together greater than the third: and the three sides are together less than the circumference of a great circle.*



Let ABC be a spherical triangle on the surface of a sphere, whose centre is O: draw the radii of the sphere OA, OB, OC to the angular points: then, since the *solid* angle at O is contained by the three *plane* angles AOB, AOC, BOC, *any two* of which are together greater than the *third*, by *Euclid* XI. 20, it follows that any two of the arcs, which measure these angles, are together greater than the third: that is, $AB+AC$ is greater than BC, $AB+BC$ greater than AC, and $AC+BC$ greater than AB.

Also, since the *solid* angle at O is contained by the three *plane* angles AOB, AOC, BOC, which are together *less than four* right angles by *Euclid* XI. 21, it is manifest that the *three* arcs AB, AC and BC are together *less* than the circumference of a great circle.

That is, $a+b>c$, $a+c>b$, $b+c>a$, and $a+b+c<2\pi$, if the radius of the sphere be 1: also, because $a+b>c$, we have $a+b+c>2c$, or $c<\frac{1}{2}(a+b+c)<\pi$: that is, any side is less than a semicircle: and since $a>c-b$, $b>a-c$ and $c>a-b$, we learn that any one side is greater than the difference of the two others.

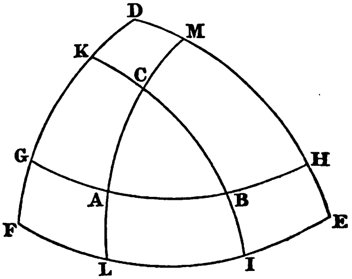
16. Since, by *Euclid* XI. 21, every solid angle is contained by plane angles which are together less than four right angles, it follows that the sum of all the sides of a *Spherical Polygon*, must be less than 2π.

In the preceding articles it is always implied that *Euclid's* definition of a solid angle is attended to, its limit being measured by a hemisphere, in the same way as his limit of a plane angle is measured by a semicircle.

THE POLAR TRIANGLE.

17. DEF. If with the angular points of a spherical triangle as poles, *great* circles of the sphere be described, the figure formed by the intersection of these circles is called the *Polar Triangle*, in contradistinction to which the original one is termed the *Primitive Triangle*.

18. *The angular points of the* polar *triangle, are the poles of the sides of the* primitive *triangle.*



Let ABC be the primitive triangle, DEF the polar triangle constructed according to the definition, and let the great circles be produced as in the Diagram:

then, since A is the pole of DE, AD is a quadrant:

and since B is the pole of DF, BD is a quadrant:

therefore DA and DB being quadrants, the point D is the pole of AB by Article (9): for the same reason, the angular points E, F, of the polar triangle are the poles of the sides AC, BC of the primitive triangle.

19. *The sides and angles of the* polar *triangle, are the supplements of the angles and sides respectively, of the* primitive *triangle.*

The same construction remaining, we have,

$$\angle A = HM = DH - DM = DH + ME - DE = \pi - DE,$$

since DH and ME are quadrants:

similarly, $\angle B = \pi - DF$, and $\angle C = \pi - EF$.

$$\text{Again, } \angle D = GH = GB + BH = GB + AH - AB$$

$$= \pi - AB, \text{ since } GB \text{ and } AH \text{ are quadrants:}$$

similarly, $\angle E = \pi - AC$, and $\angle F = \pi - BC$.

Hence, if a, b, c, A, B, C, be the sides and angles respectively of the *primitive* triangle, and a', b', c', A', B', C' those of the *polar* triangle, we shall have

$$a' = \pi - A, \quad b' = \pi - B, \quad c' = \pi - C:$$

$$A' = \pi - a, \quad B' = \pi - b, \quad C' = \pi - c:$$

and from these properties, the polar triangle is frequently called the *supplemental* triangle; but this *name* will be properly explained in Appendix II.

20. Cor. If one or more of the sides or angles of he primitive triangle, be quadrants or right angles, the corresponding angles or sides of the polar triangle, will be right angles or quadrants.

21. *The sum of the angles of a spherical triangle, ies between two and six right angles.*

For, since $a' + b' + c'$ is less than 2π, by Article (15); we shall have $A + B + C = 3\pi - (a' + b' + c')$, which is reater than π or *two* right angles: and less than 3π r *six right angles.*

A spherical triangle may therefore have *two* or *three* right angles: or, *two* or *three* obtuse angles: but *every* angle must be less than *two right* angles.

22. Cor. 1. Hence also, the sum of any two angles of a spherical triangle exceeds the third, by *less* than two right angles.

For, by Article (15), $a' + b'$ is greater than c':

$\therefore \; \pi - A + \pi - B$ is greater than $\pi - C$:

whence, π is greater than $A + B - C$, or $A + B - C$ is less than π: and similarly of the rest.

23. Cor. 2. We have, therefore, in every triangle, $a - b < c$ *à fortiori* $< \pi$, by Article (15):

$$\text{also, } A + B - C < \pi, \;\; \therefore \; A + B < \pi + C < 2\pi:$$

$$\text{and } A - B + C < \pi, \;\; \therefore \; A - B < \pi - C < \pi:$$

$$\text{whence, } \tfrac{1}{2}(a - b) < \frac{\pi}{2}, \text{ and } \tfrac{1}{2}(A - B) < \frac{\pi}{2}:$$

also, $\frac{1}{2}(a - b)$ and $\frac{1}{2}(A - B)$ being *both* less than a right angle, are said to be of the *same Affection.*

If two angular magnitudes be *both* greater or *both* less than a right angle, they are of the *same* affection; but when one of them is *greater*, and the other *less* than a right angle, they are of *different* affections.

24. If the sides of a spherical polygon be each less than a semicircle, and with its angular points as poles, great circles be described, another spherical polygon will be formed, which is *supplemental* to the former, in the sense of Article (19).

CHAPTER II.

THE INVESTIGATION OF FORMULÆ EXPRESSING THE RELATIONS BETWEEN THE SIDES AND ANGLES OF SPHERICAL TRIANGLES.

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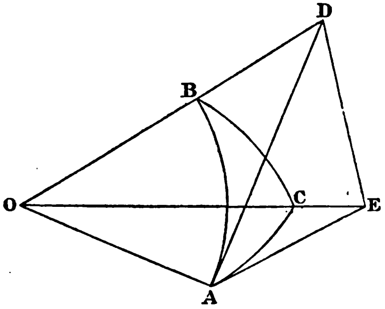
25. *To express the cosine of an angle of a spherical triangle, in terms of the sides.*



Let ABC be a triangle on the surface of a sphere whose centre is O and radius OA, the angles being A, B, C, and the corresponding opposite sides, a, b, c: draw AD, AE at right angles to OA in the planes OAB, OAC respectively, meeting OB, OC produced in D, E: join DE: then the plane angle $DAE = A$, by Article (13): now in the plane triangle DAE, we have

$$DE^2 = AE^2 + AD^2 - 2AE \,.\, AD \cos A:$$

and in the plane triangle DOE, we have

$$DE^2 = OE^2 + OD^2 - 2OE \,.\, OD \cos a,$$

since BC measures the plane angle DOE:

$$\therefore\ AE^2 + AD^2 - 2AE \,.\, AD \cos A$$
$$= OE^2 + OD^2 - 2OE \,.\, OD \cos a:$$
$$\therefore\ 2\,OE \,.\, OD \cos a$$
$$= OE^2 - AE^2 + OD^2 - AD^2 + 2AE \,.\, AD \cos A \because$$
$$= 2OA^2 + 2AE \,.\, AD \cos A:$$

$$\text{whence, } \cos a = \frac{OA}{OE} \cdot \frac{OA}{OD} + \frac{AE}{OE} \cdot \frac{AD}{OD} \cos A$$

$$= \cos b \cos c + \sin b \sin c \cos A:$$

$$\text{and } \therefore \cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

$$\text{So, } \cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c}: \ \cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}.$$

Here a side of a triangle is expressed in terms of its opposite angle and the sides which contain it: thus, $\cos a = \cos b \cos c + \sin b \sin c \cos A$, which corresponds to Article (71) of the *Plane Trigonometry:* and these *fundamental* formulæ should be committed to memory.

26. Cor. 1. If $a = b$, the expressions for $\cos A$ and $\cos B$, become identical, and therefore the angles at the base of an *isosceles* spherical triangle are equal.

If $a = b = c$, we have $\cos A = \cos B = \cos C$, and therefore every *equilateral* spherical triangle is also *equiangular.*

27. Cor. 2. If the angle at C be a right angle, we have $0 = \cos c - \cos a \cos b$, or $\cos c = \cos a \cos b$: and therefore, as $\cos a$ and $\cos b$ have the *same* or *different* signs, $\cos c$ will be *positive* or *negative:* that is, according as the sides are of the *same* or *different* affections, the *hypothenuse*, or side opposite the right angle, is *less* or *greater* than a quadrant.

28. *To express the sine of an angle of a spherical triangle, in terms of the sides.*

Since, $\sin^2 A = 1 - \cos^2 A = (1 - \cos A)(1 + \cos A)$:

$$\text{and } \cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}: \text{ we have}$$

$$1 - \cos A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{(\cos b \cos c + \sin b \sin c) - \cos a}{\sin b \sin c}$$

$$= \frac{\cos (b - c) - \cos a}{\sin b \sin c} = \frac{2 \sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(a + c - b)}{\sin b \sin c};$$

$$1+\cos A = 1+\frac{\cos a-\cos b\cos c}{\sin b\sin c}$$

$$=\frac{\cos a-(\cos b\cos c-\sin b\sin c)}{\sin b\sin c}$$

$$=\frac{\cos a-\cos(b+c)}{\sin b\sin c}=\frac{2\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(b+c-a)}{\sin b\sin c}:$$

now, if $a+b+c=2s$, as in Article (73), *Pl. Tr.*

then, $b+c-a=2(s-a)$, $a+c-b=2(s-b)$,

and $a+b-c=2(s-c)$: whence we get

$$1-\cos A=\frac{2\sin(s-b)\sin(s-c)}{\sin b\sin c}:$$

$$1+\cos A=\frac{2\sin s\sin(s-a)}{\sin b\sin c}:$$

$$\therefore \sin A=\frac{2}{\sin b\sin c}\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}.$$

The values of sin B, sin C are immediately obtained y the proper changes of symbols.

29 Cor. Since the quantity under the radical sign s symmetrical with respect to all the sides, it must be ommon to the sines of all the angles: whence, rejecting he common factors, we have

$$\sin A : \sin B : \sin C = \sin a : \sin b : \sin c;$$

r, the sines of the sides of a spherical triangle are to ne another, as the sines of the opposite angles.

30. *To express the sine, cosine and tangent of half n angle of a spherical triangle, in terms of the sides.*

From the investigation of Article (28), we find

$$\sin\tfrac{1}{2}A=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}:$$

$$\cos\tfrac{1}{2}A=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}:$$

and $\therefore \tan\frac{1}{2}A=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}.$

31. Cor. Since, $\frac{\sin \frac{1}{2} A}{\sin \frac{1}{2} B} = \sqrt{\frac{\sin a \sin (s-b)}{\sin b \sin (s-a)}}$:

it follows that $\sin \frac{1}{2} A$ is greater or less than $\sin \frac{1}{2} B$, according as a is greater or less than b: that is, $A - B$ and $a - b$ have the same algebraical sign: or, the greater side of every triangle is opposite to the greater angle: and *vice versâ.*

32. *To express the cosine of a side of a spherical triangle, in terms of the angles.*

Let a, b, c, A, B, C be the sides and angles of the proposed triangle; a', b', c', A', B', C', those of the polar triangle: then, by Article (25), we have

$$\cos A' = \frac{\cos a' - \cos b' \cos c'}{\sin b' \sin c'} .$$

$$\text{but, } \cos A' = \cos (\pi - a) = - \cos a :$$

$$\cos a' = \cos (\pi - A) = - \cos A :$$

$$\cos b' = - \cos B, \ \cos c' = - \cos C :$$

$$\sin b' = \sin (\pi - B) = \sin B, \text{ and } \sin c' = \sin C :$$

$$\text{whence, } \cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} .$$

$$\text{So, } \cos b = \frac{\cos B + \cos A \cos C}{\sin A \sin C} : \cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B} .$$

It may be proper to observe here, that the introduction of the polar triangle is necessary merely to prove that the supplements of the sides and angles will form a triangle; as Article (25) is true for every triangle.

Hence, $\cos A = \sin B \sin C \cos a - \cos B \cos C$, which expresses an angle in terms of the opposite side, and the angles adjacent to it.

33. Cor. 1. If $A = B$, we have $\cos a = \cos b$; that is, when two angles of a spherical triangle are equal, the sides opposite to them are also equal.

If $A = B = C$, we find $\cos a = \cos b = \cos c$: or every *equiangular* spherical triangle, is also equilateral.

34. Cor. 2. If C be a right angle, we have

$$\cos a = \frac{\cos A}{\sin B},$$

which is *positive* or *negative*, according as A is *less* or *greater* than a right angle: whence it follows that a will be less or greater than a quadrant, according as A is less or greater than a right angle; and *vice versâ:* that is, the sides of *right-angled* triangles are of the same affections as their opposite angles.

35. *To express the sine of a side of a spherical triangle, in terms of the angles.*

Here,
$$1 - \cos a = 1 - \frac{\cos A + \cos B \cos C}{\sin B \sin C}$$
$$= - \frac{\cos A + (\cos B \cos C - \sin B \sin C)}{\sin B \sin C}$$
$$= - \frac{\cos A + \cos (B + C)}{\sin B \sin C}$$
$$= - \frac{2 \cos \tfrac{1}{2}(A + B + C) \cos \tfrac{1}{2}(B + C - A)}{\sin B \sin C}:$$

$$1 + \cos a = 1 + \frac{\cos A + \cos B \cos C}{\sin B \sin C}$$
$$= \frac{\cos A + (\cos B \cos C + \sin B \sin C)}{\sin B \sin C}$$
$$= \frac{\cos A + \cos (B - C)}{\sin B \sin C}$$
$$= \frac{2 \cos \tfrac{1}{2}(A + B - C) \cos \tfrac{1}{2}(A + C - B)}{\sin B \sin C}:$$

whence, if we assume $A + B + C = 2S$, and

$\therefore\ B + C - A = 2(S - A),\quad A + C - B = 2(S - B),$

and $A + B - C = 2(S - C)$, we shall have

$$1 - \cos a = - \frac{2 \cos S \cos (S - A)}{\sin B \sin C}:$$

$$1 + \cos a = \frac{2 \cos (S - B) \cos (S - C)}{\sin B \sin C}:$$

$\therefore$ sin a

$$= \frac{2}{\sin B \sin C}\sqrt{-\cos S \cos (S-A) \cos (S-B) \cos (S-C)}.$$

The sines of b and c will be expressed in similar forms, the quantity affected with the radical sign being symmetrical with respect to all the angles, and therefore the same for all the sides.

36. The expression for Sin a appears in an *imaginary form,* but it gives a *real magnitude.*

For, by Article (21), $2S$ lies between π and 3π, and therefore S is intermediate in magnitude to $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$; whence cos S is essentially *negative;* also, by Article (22),

$$S - A = \tfrac{1}{2}(B + C - A)$$

is less than $\frac{1}{2}\pi$, and therefore cos $(S-A)$ is *positive:* and the same may be said of cos $(S-B)$ and cos $(S-C)$; and thus the value of sin a is real, though it appears in an imaginary form.

37. *To express the sine, cosine, and tangent of half a side of a spherical triangle, in terms of the angles.*

These are indeed included in Article (35): thus,

$$\sin \tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos (S-A)}{\sin B \sin C}}:$$

$$\cos \tfrac{1}{2}a = \sqrt{\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}}:$$

$$\text{and } \therefore \tan \tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}}:$$

to the first and last of which, the remark and explanation of the preceding article are to be applied.

38. *To express an angle of a spherical triangle, in terms of another angle, and the sides which include it.*

$$\text{Since, } \cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

$$\text{and } \sin A = \frac{\sin a}{\sin c} \sin C,$$

$$\text{we have } \cot A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \times \frac{\sin c}{\sin a \sin C}:$$

$$\therefore\ \sin C \cot A = \frac{\cos a - \cos b \cos c}{\sin a \sin b}$$

$$= \frac{\cos a - \cos b\,(\cos a \cos b + \sin a \sin b \cos C)}{\sin a \sin b}$$

$$= \frac{\cos a \sin^2 b - \sin a \sin b \cos b \cos C}{\sin a \sin b}$$

$$= \cot a \sin b - \cos b \cos C;$$

nd the angle A is therefore expressed in terms of the ngle C, and the sides a, b which include it.

39. *To express a side of a spherical triangle, in erms of another side, and the angles which are adjacent o it.*

$$\text{Since, } \cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

$$\text{and } \sin a = \frac{\sin A}{\sin C} \sin c,$$

$$\text{we have } \cot a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} \times \frac{\sin C}{\sin A \sin c}:$$

$$\therefore\ \sin c \cot a = \frac{\cos A + \cos B \cos C}{\sin A \sin B}$$

$$= \frac{\cos A + \cos B\,(\sin A \sin B \cos c - \cos A \cos B)}{\sin A \sin B}$$

$$= \frac{\cos A \sin^2 B + \sin A \sin B \cos B \cos c}{\sin A \sin B}$$

$$= \cot A \sin B + \cos B \cos c:$$

whence the side a is expressed in terms of the side c, and the angles A, B, which are adjacent to it.

This result might have been derived from the last, by means of the Polar Triangle: and, as well as the preceding one, ought to be carefully remembered.

40. *To express two angles of a spherical triangle, in erms of their opposite sides and the remaining angle.*

Generally, $\tan\frac{1}{2}(A \pm B) = \dfrac{\sin A \pm \sin B}{\cos A + \cos B}$:

but, since $\dfrac{\sin A}{\sin a} = \dfrac{\sin B}{\sin b} = \dfrac{\sin C}{\sin c}$, we have

$$(\sin A \pm \sin B)\sin c = (\sin a \pm \sin b)\sin C:$$

also, $(\cos A + \cos B)\sin c$

$$= \frac{\cos a - \cos b\cos c}{\sin b} + \frac{\cos b - \cos a\cos c}{\sin a}$$

$$= \frac{\sin a\cos a - \sin a\cos b\cos c + \sin b\cos b - \sin b\cos a\cos c}{\sin a\sin b}$$

$$= \frac{\frac{1}{2}(\sin 2a + \sin 2b) - \cos c\sin(a+b)}{\sin a\sin b}$$

$$= \frac{\sin(a+b)\cos(a-b) - \cos c\sin(a+b)}{\sin a\sin b}$$

$$= \sin(a+b)\left\{\frac{\cos(a-b) - \cos c}{\sin a\sin b}\right\}$$

$$= \sin(a+b)\left\{\frac{\cos a\cos b + \sin a\sin b - \cos c}{\sin a\sin b}\right\}$$

$$= \sin(a+b)\left\{1 - \frac{\cos c - \cos a\cos b}{\sin a\sin b}\right\}$$

$$= \sin(a+b)\{1 - \cos C\}:$$

whence, we have $\tan\frac{1}{2}(A \pm B) = \dfrac{\sin a \pm \sin b}{\sin(a+b)}\,\dfrac{\sin C}{1 - \cos C}$:

and the simplest reductions now give

$$\tan\tfrac{1}{2}(A+B) = \frac{2\sin\frac{1}{2}(a+b)\cos\frac{1}{2}(a-b)}{2\sin\frac{1}{2}(a+b)\cos\frac{1}{2}(a+b)}\,\frac{2\sin\frac{1}{2}C\cos\frac{1}{2}C}{2\sin^2\frac{1}{2}C}$$

$$= \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C:$$

$$\tan\tfrac{1}{2}(A-B) = \frac{2\cos\frac{1}{2}(a+b)\sin\frac{1}{2}(a-b)}{2\cos\frac{1}{2}(a+b)\sin\frac{1}{2}(a+b)}\,\frac{2\sin\frac{1}{2}C\cos\frac{1}{2}C}{2\sin^2\frac{1}{2}C}$$

$$= \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C.$$

These simple and elegant formulæ, for finding the *sum and* difference of A and B, were discovered by

Lord Napier, and are termed *Napier's* first and second *Analogies*.

41. *To express two sides of a spherical triangle, in terms of their opposite angles, and the remaining side.*

Retaining the notation hitherto used, we have, in the polar triangle, by the last Article,

$$\tan \tfrac{1}{2}(A'+B') = \frac{\cos \frac{1}{2}(a'-b')}{\cos \frac{1}{2}(a'+b')} \cot \tfrac{1}{2}C' :$$

$$\tan \tfrac{1}{2}(A'-B') = \frac{\sin \frac{1}{2}(a'-b')}{\sin \frac{1}{2}(a'+b')} \cot \tfrac{1}{2}C' :$$

and by making the proper substitutions, as in Article (32), we shall find

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \tfrac{1}{2}c :$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \tfrac{1}{2}c :$$

which are termed *Napier's* third and fourth *Analogies*.

These results might have been established in a manner precisely similar to the preceding.

42. Cor. From the first of these Analogies,

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}C,$$

it appears that $\tan \frac{1}{2}(A+B)$ has the same algebraical sign as $\cos \frac{1}{2}(a+b)$, because $\frac{1}{2}(a-b)$ and $\frac{1}{2}C$ are both less than $\frac{1}{2}\pi$, by Articles (23) and (21), and therefore both $\cos \frac{1}{2}(a-b)$ and $\cot \frac{1}{2}C$ are positive: whence it follows that

$$\tfrac{1}{2}(A+B) \text{ and } \tfrac{1}{2}(a+b)$$

are always of the same affection.

The preceding articles comprise all the formulæ necessary for the solution of spherical triangles, which will be treated of in the next chapter: and their adaptation to logarithmic computation will be there explained.

43. Since

$$\sin\tfrac{1}{2}(A+B)=\sin\tfrac{1}{2}A\cos\tfrac{1}{2}B+\cos\tfrac{1}{2}A\sin\tfrac{1}{2}B$$

$$=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}\sqrt{\frac{\sin s\sin(s-b)}{\sin a\sin c}}$$

$$+\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}\sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin a\sin c}}$$

$$=\frac{\sin(s-b)+\sin(s-a)}{\sin c}\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}$$

$$=\frac{\sin\frac{1}{2}(2s-a-b)\cos\frac{1}{2}(a-b)}{\sin\frac{1}{2}c\cos\frac{1}{2}c}\cos\tfrac{1}{2}C$$

$$=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}\cos\tfrac{1}{2}C:$$ we shall have

$$\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c=\cos\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C:$$

similarly, $\cos\frac{1}{2}(A+B)\cos\frac{1}{2}c=\cos\frac{1}{2}(a+b)\sin\frac{1}{2}C$:

whence, $\tan\frac{1}{2}(A+B)=\dfrac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{1}{2}C$.

In the same way it may be proved that

$$\sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c=\sin\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C,$$

$$\cos\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c=\sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C:$$

whence, $\tan\frac{1}{2}(A-B)=\dfrac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{1}{2}C$:

and thus the first and second Analogies are derived from Article (30).

CHAPTER III.

THE SOLUTION OF SPHERICAL TRIANGLES.

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44. In every spherical triangle, the three sides and he three angles are connected by *six* equations of the orms :

$$\text{os } A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}, \text{ and } \cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} :$$

nd since, by the proper substitutions, they would give *hree* independent equations involving *six* different quantities, it follows that if three of these quantities be given, he remaining three may generally be found.

If one of the parts of the triangle be a *right angle*, r a *quadrant*, it is manifest that only *two* additional arts will be required to determine the rest : and on this ccount, the solutions of spherical triangles are distriuted under the three following heads :

(1) Right-angled Triangles :
(2) Quadrantal Triangles :
(3) Oblique-angled Triangles :

nd proper applications of the preceding theorems will effect the solutions of all their cases.

I. *Right-angled Triangles.*

45. Let *ABC* be a spherical triangle, right-angled at *C*: then, by means of the fundamental formulæ above noticed, if any *two* of its parts be given, the rest may be found (as will be shewn in the next article but one) : but the substitutions and eliminations necessary to effect this, would in many cases be too tedious for practice, especially to those who are not familiar with the management of algebraical formulæ, and their results would be burdensome to the memory.

These circumstances suggested to Lord Napier the *invention of his celebrated System of* Circular Parts, and the two easily remembered *General Rules* immediately connected with it, which we will now endeavour to explain and prove.

Statement of Napier's Rules.

46. If C be supposed to be the right angle, there remain *five* other parts belonging to every triangle; the two *sides* or *legs* a, b: the *hypothenuse* c, and the two *angles*, A, B.

Now the two sides a, b, the *complement* of the hypothenuse $\frac{1}{2}\pi - c$, and the *complements* of the two angles $\frac{1}{2}\pi - A$ and $\frac{1}{2}\pi - B$, are by Napier termed *Circular Parts*, the right angle being left entirely out of the consideration.

Then, if any one of these be assumed to be what he calls a *Middle Part*, the two parts adjoining it close on each side, are called the *Adjacent Extremes*, and the two remaining parts which are separated from it by an adjacent part, are termed the *Opposite Extremes*.

This being premised, the two following equalities are found universally to obtain, and are known by the name of *Napier's Rules* for the Solution of right-angled Spherical Triangles:

(1) The *Sine* of the Middle Part

= the Product of the *Tangents* of the adjacent extremes.

(2) The *Sine* of the Middle Part

= the Product of the *Cosines* of the opposite extremes:

from which, if any two of the parts employed be given, the remaining part may be found.

47. *Demonstrations of Napier's Rules.*

I. Let a be the *middle part*: then $\frac{1}{2}\pi - B$ and b are the *adjacent extremes*: and $\frac{1}{2}\pi - A$ and $\frac{1}{2}\pi - c$ the *opposite extremes*: and it is required to prove that

$$\sin a = \tan\left(\tfrac{1}{2}\pi - B\right)\tan b$$
$$= \cos\left(\tfrac{1}{2}\pi - A\right)\cos\left(\tfrac{1}{2}\pi - c\right).$$

$$\text{Since, } \cos b = \frac{\cos B + \cos A \cos C}{\sin A \sin C} = \frac{\cos B}{\sin A},$$

or, $\sin A = \dfrac{\cos B}{\cos b}$; and also $\sin A = \dfrac{\sin a}{\sin b} \sin B$:

we have $\dfrac{\sin a}{\sin b} \sin B = \dfrac{\cos B}{\cos b}$, which gives

$$\therefore \sin a = \frac{\cos B}{\sin B} \frac{\sin b}{\cos b} = \cot B \tan b$$

$$= \tan\left(\tfrac{1}{2}\pi - B\right) \tan b \ldots\ldots\ldots\ldots\ldots (1).$$

gain, since $\dfrac{\sin a}{\sin c} = \dfrac{\sin A}{\sin C} = \sin A$: or, $\sin a = \sin A \sin c$:

We have $\sin a = \cos\left(\tfrac{1}{2}\pi - A\right) \cos\left(\tfrac{1}{2}\pi - c\right) \ldots\ldots\ldots (2)$.

Exactly in the same way, it may be shewn that

$$\sin b = \tan\left(\tfrac{1}{2}\pi - A\right) \tan a \ldots\ldots\ldots\ldots\ldots (3).$$

$$= \cos\left(\tfrac{1}{2}\pi - B\right) \cos\left(\tfrac{1}{2}\pi - c\right) \ldots\ldots\ldots (4).$$

II. Let $\frac{1}{2}\pi - c$ be the *middle part:* then $\frac{1}{2}\pi - A$ and $\frac{1}{2}\pi - B$ are the *adjacent extremes:* and a and b the *opposite extremes:* and the equalities to be proved are

$$\sin\left(\tfrac{1}{2}\pi - c\right) = \tan\left(\tfrac{1}{2}\pi - A\right) \tan\left(\tfrac{1}{2}\pi - B\right)$$

$$= \cos a \cos b.$$

Here, $\cos c = \dfrac{\cos C + \cos A \cos B}{\sin A \sin B} = \cot A \cot B$:

or, $\sin\left(\tfrac{1}{2}\pi - c\right) = \tan\left(\tfrac{1}{2}\pi - A\right) \tan\left(\tfrac{1}{2}\pi - B\right) \ldots\ldots (5)$.

Also, $\cos C = \dfrac{\cos c - \cos a \cos b}{\sin a \sin b} = 0$, and $\therefore \cos c = \cos a \cos b$:

$$\text{or, } \sin\left(\tfrac{1}{2}\pi - c\right) = \cos a \cos b \ldots\ldots\ldots\ldots\ldots (6).$$

III. Let $\frac{1}{2}\pi - A$ be the *middle part:* then b and $\frac{1}{2}\pi - c$ are the *adjacent extremes:* and a and $\frac{1}{2}\pi - B$ the *opposite extremes:* and we have to shew that

$$\sin\left(\tfrac{1}{2}\pi - A\right) = \tan b \tan\left(\tfrac{1}{2}\pi - c\right)$$

$$= \cos a \cos\left(\tfrac{1}{2}\pi - B\right).$$

Since, $\cos a \cos b = \cos c$, from (6), we have

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\cos a \cos b - \cos^2 b \cos c}{\sin b \cos b \sin c}$$

$$=\frac{\cos c-\cos^2 b\cos c}{\sin b\cos b\sin c}=\frac{(1-\cos^2 b)\cos c}{\sin b\cos b\sin c}=\frac{\sin^2 b\cos c}{\sin b\cos b\sin c}$$

$$=\frac{\sin b}{\cos b}\,\frac{\cos c}{\sin c}=\tan b\cot c:$$

$$\text{or, } \sin(\tfrac{1}{2}\pi-A)=\tan b\tan(\tfrac{1}{2}\pi-c)\,\ldots\ldots\ldots(7).$$

Also, since $\cos a=\dfrac{\cos A+\cos B\cos C}{\sin B\sin C}=\dfrac{\cos A}{\sin B}$,

we have $\cos A=\cos a\sin B$:

$$\text{or, } \sin(\tfrac{1}{2}\pi-A)=\cos a\cos(\tfrac{1}{2}\pi-B)\,\ldots\ldots\ldots(8).$$

Precisely in the same manner, it may be proved that

$$\sin(\tfrac{1}{2}\pi-B)=\tan a\tan(\tfrac{1}{2}\pi-c)\ldots\ldots\ldots(9).$$

$$=\cos b\cos(\tfrac{1}{2}\pi-A)\ldots\ldots\ldots(10).$$

The *ten* formulæ here investigated comprise all the cases that can possibly occur, since out of *five* things taken *two* together, there can be formed

$$\frac{5\,.\,4}{1\,.\,2}=10$$

different combinations; and they are all adapted to logarithmic computation.

48. Some writers consider the employment of these Rules as *unscientific*, and as having a tendency to render the knowledge of the subject both *confined* and *technical*, and suggest that every instance should be worked out when it occurs, as has been done above. This may be the case, when they are viewed only in a *theoretical* light: but their *practical* importance in the hands of persons, who cannot be expected to be very familiar with the transformation of Trigonometrical formulæ, seems to shew that an *Elementary Treatise* upon the Subject, would be imperfect without them.

Examples.

(1) Given a and b, to find the rest.

From (1), $\sin a=\cot B\tan b$, $\therefore\ \cot B=\sin a\cot b$:

(3), $\sin b=\cot A\tan a$, $\therefore\ \cot A=\sin b\cot a$:

(6), $\cos c=\cos a\cos b$:

whence B, A, c may be found: and it is manifest that there is no ambiguity in the solutions, because none of the quantities required are to be found from their *sines*. See Article (32), *Pl. Tr.*

(2) Given a and c, to find the rest.

From (2), $\sin a = \sin A \sin c$, $\therefore \sin A = \frac{\sin a}{\sin c}$:

(6), $\cos c = \cos a \cos b$, $\therefore \cos b = \frac{\cos c}{\cos a}$:

(9), $\cos B = \tan a \cot c$:

and there can be no ambiguity except in the value of A: but this is removed by the consideration that A and a are always of the same affection. See Article (34).

(3) Given A and B, to find the rest.

From (8), $\cos A = \cos a \sin B$, $\therefore \cos a = \frac{\cos A}{\sin B}$:

(10), $\cos B = \cos b \sin A$, $\therefore \cos b = \frac{\cos B}{\sin A}$:

(5), $\cos c = \cot A \cot B$:

and they are all found without ambiguity.

(4) Given a and B, to find the rest.

From (1), $\sin a = \cot B \tan b$, $\therefore \tan b = \sin a \tan B$:

(9), $\cos B = \tan a \cot c$, $\therefore \cot c = \cot a \cos B$:

(8), $\cos A = \cos a \sin b$:

and there is no ambiguity in the solution.

(5) Given c and A, to find the rest.

From (5), $\cos c = \cot A \cot B$, $\therefore \cot B = \tan A \cos c$:

(7), $\cos A = \tan b \cot c$, $\therefore \tan b = \cos A \tan c$:

(2), $\sin a = \sin A \sin c$:

and the apparent ambiguity in the last, is removed by considering that a and A are always of the same affection.

(6) Given a and A, to find the rest.

From (2), $\sin a = \sin c \sin A$, $\therefore \sin c = \frac{\sin a}{\sin A}$:

$$(8),\ \cos A = \cos a \sin B,\ \therefore \sin B = \frac{\cos A}{\cos a}:$$

$$(3),\ \sin b = \tan a \cot A:$$

where there is nothing to decide whether c, B, b should be greater or less than $\frac{1}{2}\pi$, and consequently the solution remains ambiguous: also, it is the only one that occurs in right-angled spherical triangles.

The circumstances of this solution may be easily explained; for, if two great circles cut one another at A and D, it is evident that two equal arcs BC may be drawn from one of them at right angles to the other, equally distant from the points A and D: and thus, both the triangles ABC will have the proposed data, and the parts required will be supplemental to each other.

49. Whenever the function by means of which any part of a triangle is to be determined, is nearly equal to its extreme values, recourse must be had to transformations of the formulæ above given, in order to avoid it.

Thus, if c be small, instead of the formula,

$$\cos c = \cot A \cot B,$$

we should use $\tan^2 \frac{1}{2} c = -\dfrac{\cos(A+B)}{\cos(A-B)}$,

which is derived from it by the following process:

$$2 \sin^2 \tfrac{1}{2} c = 1 - \cos c = 1 - \cot A \cot B$$

$$= \frac{\sin A \sin B - \cos A \cos B}{\sin A \sin B} = -\frac{\cos(A+B)}{\sin A \sin B}:$$

$$2 \cos^2 \tfrac{1}{2} c = 1 + \cos c = 1 + \cot A \cot B$$

$$= \frac{\sin A \sin B + \cos A \cos B}{\sin A \sin B} = \frac{\cos(A-B)}{\sin A \sin B}:$$

$$\text{and } \therefore \tan^2 \tfrac{1}{2} c = -\frac{\cos(A+B)}{\cos(A-B)}:$$

wherein $\cos(A+B)$ is essentially *negative* by Article (21), since $A+B$ is greater than $\pi - C$, or $\frac{1}{2}\pi$.

Again, if a be nearly equal to $\frac{1}{2}\pi$, we should not *employ* the equation $\sin a = \sin A \sin c$, to determine it.

For, we have $\tan^2(45 - \frac{1}{2}a) = \dfrac{1 - \sin a}{1 + \sin a}$

$$= \frac{1 - \sin c \sin A}{1 + \sin c \sin A} = \frac{1 - \tan\theta}{1 + \tan\theta}, \text{ suppose:}$$

$$\therefore \tan^2(45 - \tfrac{1}{2}a) = \tan(45^\circ - \theta),$$

from which a may be conveniently obtained, as appears from Article (192), *Pl. Tr.*

Also, if B be small, from $\cos B = \tan a \cot c$, we have

$$\tan^2 \tfrac{1}{2} B = \frac{1 - \cos B}{1 + \cos B} = \frac{1 - \tan a \cot c}{1 + \tan a \cot c} = \frac{\sin(c - a)}{\sin(c + a)}:$$

which may be used for the same reasons.

II. *Quadrantal Triangles.*

50. Let ABC be a spherical triangle, whereof the side $AB = c$ is a quadrant: then, if the polar triangle be constructed, we shall have a right-angled triangle whose sides are $\pi - A$, $\pi - B$, $\pi - C$, and angles $\pi - a$, $\pi - b$ and $\frac{1}{2}\pi$, the right angle being opposite to the side $\pi - C$: whence, if to this triangle, *Napier's* Rules above given be applied, the circular parts must evidently be the two sides,

$$\text{or, } \pi - A \text{ and } \pi - B:$$

the complement of the hypothenuse,

$$\text{or, } \tfrac{1}{2}\pi - (\pi - C) = C - \tfrac{1}{2}\pi:$$

and the complements of the two angles,

$$\text{or, } \tfrac{1}{2}\pi - (\pi - a) = a - \tfrac{1}{2}\pi, \text{ and } \tfrac{1}{2}\pi - (\pi - b) = b - \tfrac{1}{2}\pi:$$

and consequently these Rules will be sufficient for determining the relations between the different parts of the quadrantal triangle, if the *Circular Parts* be the *Supplements* of the two *Angles:* and the *Complements* of the *hypothenusal angle* and of the two *sides*, taken *negatively:* the quadrantal side being left entirely out of the consideration in deciding whether the *extremes* are *adjacent* or *opposite.*

51. If necessary, the demonstration of each individual *case may easily* be derived from the fundamental

formulæ of the preceding chapter: and it will make no difference if the *angles themselves*, the complements of the *two sides*, and that of the *hypothenusal angle* taken *negatively*, be considered the circular parts.

III. *Oblique-angled Triangles.*

52. In every oblique-angled spherical triangle, there are six distinct parts, of which three being given, the remaining three may generally be found; and a little consideration will make it appear that all the solutions, essentially differing from each other, are comprised in the following articles.

53. CASE 1. *Given two sides* a, b, *and the angle* A, *opposite one of them: to find the rest.*

Here, $\frac{\sin B}{\sin A} = \frac{\sin b}{\sin a}$, and $\therefore \sin B = \frac{\sin b}{\sin a} \sin A$:

and from *Napier's* Analogies, we have

$$\cot \tfrac{1}{2} C = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(A+B):$$

$$\tan \tfrac{1}{2} c = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a+b):$$

which are all adapted to logarithmic computation.

54. *To find* C *and* c, *independently of* B.

We have from Article (38),

$$\cot A \sin C = \cot a \sin b - \cos b \cos C:$$

$$\therefore \sin C + \tan A \cos b \cos C = \cot a \sin b \tan A:$$

whence, agreeably to Ex. 9, page 84, assuming

$\tan \theta = \tan A \cos b$, we shall have

$$\sin C + \tan \theta \cos C = \cot a \sin b \tan A:$$

$$\text{or, } \frac{\sin (C+\theta)}{\cos \theta} = \cot a \sin b \tan A:$$

$$\text{or, } \sin (C+\theta) = \cot a \sin b \tan A \cos \theta$$

$$= \frac{\cot a \sin b \tan \theta \cos \theta}{\cos b} = \cot a \tan b \sin \theta:$$

from which $C + \theta$, and therefore C becomes known.

Again, from Article (25), we have

$$\sin b \sin c \cos A + \cos b \cos c = \cos a:$$

$$\text{or, } \cos A \tan b \sin c + \cos c = \frac{\cos a}{\cos b}:$$

hence, if $\tan\theta = \cos A \tan b$, we obtain

$$\sin c \tan\theta + \cos c = \frac{\cos a}{\cos b}:$$

$$\text{or, } \cos(c-\theta) = \frac{\cos a}{\cos b}\cos\theta:$$

hich, from the known value of θ, determines the value f c in a logarithmic form.

55. Case 2. *Given two angles* A, B, *and the side* a, *posite one of them: to find the rest.*

$$\text{Here, } \frac{\sin b}{\sin a} = \frac{\sin B}{\sin A}, \text{ and } \therefore \sin b = \frac{\sin B}{\sin A}\sin a:$$

so, by means of *Napier's* Analogies, we have

$$\tan\tfrac{1}{2}c = \frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}\tan\tfrac{1}{2}(a+b):$$

$$\cot\tfrac{1}{2}C = \frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}\tan\tfrac{1}{2}(A+B):$$

r, instead of the formula for finding C, we have

$$\sin C = \frac{\sin c}{\sin a}\sin A = \frac{\sin c}{\sin b}\sin B.$$

56. *To find* C *and* c, *without the previous determin-tion of* b.

Since, $\cos a \sin B \sin C - \cos B \cos C = \cos A$:

$$\therefore \cos a \tan B \sin C - \cos C = \frac{\cos A}{\cos B}:$$

d, assuming $\cot\theta = \cos a \tan B$, as before, we get

$$\sin(C-\theta) = \frac{\cos A}{\cos B}\sin\theta, \text{ from which C is found.}$$

Again, from Article (39), we have

$$\cot a \sin c - \cos B \cos c = \cot A \sin B:$$

or, $\sin c - \tan a \cos B \cos c = \tan a \cot A \sin B$:

whence, if $\tan \theta = \tan a \cos B$, we obtain

$$\sin (c - \theta) = \tan a \cot A \sin B \cos \theta = \cot A \tan B \sin \theta:$$

from which c may be found.

57. Case 3. *Given two sides* a, b, *and the included angle* C: *to find the rest.*

Here, $\tan \frac{1}{2} (A + B) = \dfrac{\cos \frac{1}{2} (a - b)}{\cos \frac{1}{2} (a + b)} \cot \frac{1}{2} C$:

$$\tan \tfrac{1}{2} (A - B) = \frac{\sin \frac{1}{2} (a - b)}{\sin \frac{1}{2} (a + b)} \cot \tfrac{1}{2} C:$$

from which, $\frac{1}{2}(A + B)$ and $\frac{1}{2}(A - B)$ are found:

and thence, $\left.\begin{matrix} A = \frac{1}{2}(A + B) + \frac{1}{2}(A - B) \\ B = \frac{1}{2}(A + B) - \frac{1}{2}(A - B) \end{matrix}\right\}$ are determined:

also, $\sin c = \dfrac{\sin C}{\sin A} \sin a = \dfrac{\sin C}{\sin B} \sin b.$

58. *To find* c, *without involving the values of* A *and* B.

Since, $\cos c = \cos a \cos b + \sin a \sin b \cos C$,

this might be adapted to logarithms at once: but it is for some purposes convenient to determine it in another form: thus,

$$\cos c = \cos a \cos b + \sin a \sin b \,(1 - \operatorname{vers} C)$$
$$= \cos (a - b) - \sin a \sin b \operatorname{vers} C:$$

whence, subtracting each member of the equality from 1, we have

$$\operatorname{vers} c = \operatorname{vers} (a - b) + \sin a \sin b \operatorname{vers} C$$
$$= \operatorname{vers} (a - b) \left\{ 1 + \frac{\sin a \sin b \operatorname{vers} C}{\operatorname{vers} (a - b)} \right\}:$$

and if $\tan^2 \theta = \dfrac{\sin a \sin b \operatorname{vers} C}{\operatorname{vers} (a - b)}$, this becomes

$\operatorname{vers} c = \operatorname{vers} (a - b) \sec^2 \theta$, from which c is found.

59. CASE 4. *Given two angles* A, B, *and the adjacent side* c: *to find the rest.*

Here, $\tan\frac{1}{2}(a+b)=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{1}{2}c$:

$$\tan\frac{1}{2}(a-b)=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{1}{2}c:$$

and thence, $\left.\begin{array}{l} a=\frac{1}{2}(a+b)+\frac{1}{2}(a-b) \\ b=\frac{1}{2}(a+b)-{}_{2}(a-b)\end{array}\right\}$ are determined:

$$\text{also, } \sin C=\frac{\sin c}{\sin a}\sin A=\frac{\sin c}{\sin b}\sin B.$$

60. *To find* C, *by an independent process.*

Since, $\cos C=\sin A\sin B\cos c-\cos A\cos B$

$$=\sin A\sin B(1-\text{vers}\,c)-\cos A\cos B$$

$$=\sin A\sin B-\cos A\cos B-\sin A\sin B\,\text{vers}\,c:$$

$$\therefore 2\sin^2\tfrac{1}{2}C=2\cos^2\tfrac{1}{2}(A+B)+\sin A\sin B\,\text{vers}\,c$$

$$=2\cos^2\tfrac{1}{2}(A+B)\left\{1+\frac{\sin A\sin B\,\text{vers}\,c}{2\cos^2\frac{1}{2}(A+B)}\right\}$$

$=2\cos^2\frac{1}{2}(A+B)\sec^2\theta$, if we assume

$$\tan^2\theta=\frac{\sin A\sin B\,\text{vers}\,c}{2\cos^2\frac{1}{2}(A+B)}:$$

and thus the value of C is found.

The same may however be effected in another form: For,

$$2\cos^2\tfrac{1}{2}C-1=(1-2\sin^2\tfrac{1}{2}c)\sin A\sin B-\cos A\cos B$$

$$=-\cos(A+B)-2\sin^2\tfrac{1}{2}c\sin A\sin B:$$

$$\therefore \cos^2\tfrac{1}{2}C=\sin^2\tfrac{1}{2}(A+B)-\sin^2\tfrac{1}{2}c\sin A\sin B:$$

whereof the latter term $\sin^2\frac{1}{2}c\sin A\sin B$, being less than 1, may be assumed $=\sin^2\theta$: and thus we have

$$\cos^2\tfrac{1}{2}C=\sin^2\tfrac{1}{2}(A+B)-\sin^2\theta$$

$=\sin\{\frac{1}{2}(A+B)+\theta\}\sin\{\frac{1}{2}(A+B)-\theta\}$, by Article (42), *Pl. Tr., which is in a form* proper for computation.

61. Case 5. *Given the three sides* a, b, c: to *find the angles.*

Here, we have four formulæ for determining an angle:

$$\sin A = \frac{2}{\sin b \sin c} \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}:$$

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}}:$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}}:$$

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}}:$$

and to which of these solutions, the preference is to be given, must be decided according to the observations made in Article (108), *Pl. Tr.*

62. Case 6. *Given the three angles,* A, B, C: *to find the sides.*

A side may be found by any one of the four formulæ:

$$\sin a = \frac{2}{\sin B \sin C} \sqrt{-\cos S \cos (S-A) \cos (S-B) \cos (S-C)}:$$

$$\sin \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos (S-A)}{\sin B \sin C}}:$$

$$\cos \tfrac{1}{2} a = \sqrt{\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}}:$$

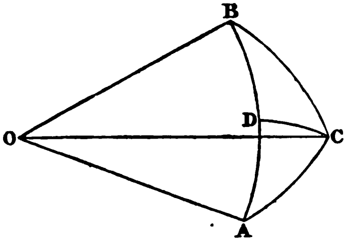
$$\tan \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}}:$$

and the eligibility of any one of these solutions, in preference to the rest, will depend upon the Article last referred to.

63. By drawing an arc of a great circle from one of the angles, perpendicular to the opposite side, *Napier's* Rules for the solution of right-angled triangles, will be rendered sufficient for the solution of all spherical

triangles whatsoever. For brevity's sake, we will exemplify their application in the following cases only.

(1) Let A, B, a be given to find the rest: and suppose the arc CD of a great circle to be drawn perpendicular to AB: then if $BD = \theta$ and $\therefore AD = c - \theta$, we have by *Napier's* Rules:



$$\cos B = \tan\theta \cot a, \text{ and } \therefore \tan\theta = \tan a \cos B,$$

and thus θ is found:

also, $\sin\theta = \cot B \tan CD$, and $\sin(c - \theta) = \cot A \tan CD$,

$$\therefore \frac{\sin(c-\theta)}{\sin\theta} = \cot A \tan B,$$

$$\text{and } \sin(c-\theta) = \cot A \tan B \sin\theta,$$

whence the value of $c - \theta$ becomes known: and thus in both the right-angled triangles CDA, CDB we have every thing requisite for a complete solution.

If we consider this in connection with Article (56), we shall see that the introduction of the subsidiary angle θ amounts to the same thing as letting fall the perpendicular CD upon the side AB, the results in the two cases being precisely the same. Similarly in other cases.

(2) Let a, b, c be given, to find the three angles: then, by *Napier's* Rules, we have

$$\cos a = \cos BD \cos CD, \text{ and } \cos b = \cos AD \cos CD:$$

$$\text{whence, } \frac{\cos a}{\cos b} = \frac{\cos BD}{\cos AD}:$$

or, *the cosines of the segments of the base, are proportional to the cosines of the adjacent sides*:

$$\text{also, } \frac{\cos a - \cos b}{\cos a + \cos b} = \frac{\cos BD - \cos AD}{\cos BD + \cos AD}$$

$$= \frac{\sin \frac{1}{2}(AD + BD) \sin \frac{1}{2}(AD - BD)}{\cos \frac{1}{2}(AD + BD) \cos \frac{1}{2}(AD - BD)}$$

$$= \tan \tfrac{1}{2} c \tan \tfrac{1}{2}(AD - BD),$$

from which $\frac{1}{2}(AD - BD)$ is found: and $\frac{1}{2}(AD + BD)$ being given, two sides in each of the right-angled triangles ACD, BCD become known, and consequently all the angles of the triangle may be determined.

Also, CD may be obtained from either of the equations

$$\cos CD = \frac{\cos a}{\cos BD} = \frac{\cos b}{\cos AD}.$$

(3) Let A, B, C be given, to find the rest; then, as before,

$$\cos A = \cos CD \sin ACD, \quad \cos B = \cos CD \sin BCD:$$

$$\text{whence, } \frac{\cos A}{\cos B} = \frac{\sin ACD}{\sin BCD};$$

or, *the sines of the segments of the vertical angle, are proportional to the cosines of the corresponding angles at the base:* therefore, if these segments be called α, β, we have

$$\frac{\cos A - \cos B}{\cos A + \cos B} = \frac{\sin \alpha - \sin \beta}{\sin \alpha + \sin \beta}$$

$$= \frac{\tan \frac{1}{2}(\alpha - \beta)}{\tan \frac{1}{2}(\alpha + \beta)} = \cot \tfrac{1}{2} C \tan \tfrac{1}{2}(\alpha - \beta):$$

from which $\frac{1}{2}(\alpha - \beta)$ being found, and $\frac{1}{2}(\alpha + \beta)$, or $\frac{1}{2}C$ being given, the segments α, β, are easily determined: and thus, in each of the right-angled triangles ACD, BCD, two angles being known, the sides of the proposed triangle may be ascertained.

From this, CD may be found by either of the equations

$$\cos CD = \frac{\cos A}{\sin \alpha} = \frac{\cos B}{\sin \beta}.$$

64. In the preceding articles, no notice has been taken of any ambiguity that may occur in the solutions,

when the required part is to be determined by means of ts sine: and it will now be shewn under what circumstances a complete solution may be obtained or not.

If a, b, A be given, to find the angle B: then, by Article (42), $A+B$ is greater or less than π, according s $a+b$ is greater or less than π.

(1) Let $a+b$ be $>\pi$; then

$$A+B \text{ is } >\pi, \text{ and } \therefore B \text{ is } >\pi - A.$$

If b be $>a$, by Article (31), B is $>A$:

$$\text{let } A \text{ be } <\tfrac{1}{2}\pi, \ \therefore\ B \text{ is } >\tfrac{1}{2}\pi > A,$$

nd B will therefore be *determined:*

$$\text{let } A \text{ be } >\tfrac{1}{2}\pi, \ \therefore\ B \text{ is } >\tfrac{1}{2}\pi > A, \text{ or } <\tfrac{1}{2}\pi < A,$$

nd there *may be* a solution or not, dependent upon the *bsolute* magnitudes of the *data.*

If b be $<a$, therefore B is $<A$:

$$\text{let } A \text{ be } <\tfrac{1}{2}\pi, \ \therefore\ B \text{ is } >\tfrac{1}{2}\pi > A,$$

nd consequently there is *no possible* solution:

$$\text{let } A \text{ be } >\tfrac{1}{2}\pi, \ \therefore\ B \text{ is } <\tfrac{1}{2}\pi, \text{ or } >\tfrac{1}{2}\pi,$$

nd the solution is *ambiguous.*

(2) Let $a+b$ be $<\pi$; then

$$A+B \text{ is } <\pi, \text{ and } \therefore B \text{ is } <\pi - A.$$

If b be $>a$, therefore B is $>A$:

$$\text{let } A \text{ be } <\tfrac{1}{2}\pi, \ \therefore\ B \text{ is } <\tfrac{1}{2}\pi, \text{ or } >\tfrac{1}{2}\pi,$$

nd the solution is *ambiguous:*

$$\text{let } A \text{ be } >\tfrac{1}{2}\pi, \ \therefore\ B \text{ is } <\tfrac{1}{2}\pi < A,$$

nd consequently there is *no possible* solution.

If b be $<a$, then B is $<A$:

$$\text{let } A \text{ be } <\tfrac{1}{2}\pi, \ \therefore\ B \text{ is } <\tfrac{1}{2}\pi < A, \text{ or } >\tfrac{1}{2}\pi > A,$$

nd there *may be* a solution or not, as before:

$$\text{let } A \text{ be } >\tfrac{1}{2}\pi, \ \therefore\ B \text{ is } <\tfrac{1}{2}\pi < A,$$

nd *B will be fully determined.*

(3) Let $a + b = \pi$; then will $A + B = \pi$:

and $\therefore\ B = \pi - A$ is immediately found.

If A, B, a, be the given quantities, precisely the same mode of reasoning may be applied to determine the circumstances of the solution, and it may be materially assisted by proper diagrams.

65. Cor. From the preceding article, it appears that certain *limitations* in the *data*, will sometimes be requisite to ensure a *possible* solution, and therefore the existence of a *real* triangle.

Some of these have just been pointed out, and in most cases they are easily determined; as in the instance we have been considering, if instead of B, we find $\sin C$ from the expression

$$\cot A \sin C + \cos b \cos C = \cot a \sin b,$$

by means of a quadratic equation, which will evidently be of the form

$$mx^2 + nx + r = 0:$$

when there are *two positive* values of x, there will be *two triangles*, and the solution will be *ambiguous*: when there is only *one positive* value of x, there will be only *one triangle*, and the solution will be *complete*: and when both the values of x are *negative* or *imaginary*, there will be *no triangle* formed, in consequence of the *incongruity* of the *data* for that purpose.

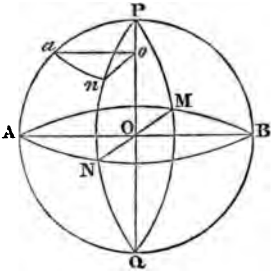
Many additional formulæ connected with the preceding chapters will be found in the Second Appendix.

CHAPTER IV.

THE AREA OF A SPHERICAL TRIANGLE, AND THE COMPARISON OF SOLID ANGLES.

I. *The Area of a Spherical Triangle.*

66. *The surface of a spherical Lune, is proportional the angle contained between the planes of the semicircles which it is formed.*



Let the semicircles *PAQ, PNQ,* on the surface of the phere whose radius is $OA = ON$, intersect each other in he diameter *POQ*: then the surface *APNQ* included etween them is called a *Lune:* and it is obvious that f any multiple whatever of the arc *AN*, which, by Article 14), measures the angle *APN*, be taken, the surface of he corresponding lune will be the same multiple of that f the lune *APNQ*: that is, the surface of the lune *PNQ* is *proportional* to the arc *AN*, or to the angle *PN*.

Hence, if $\angle APN = A^\circ$, and the radius of the sphere $= r$, so that its whole surface $= 4\pi r^2$: (see Article (23) f the *Differential Calculus*): then,

$$\frac{\text{area of } APNQ}{4\pi r^2} = \frac{A^\circ}{360^\circ} = \frac{A}{360}:$$

$$\therefore \text{ the area of } APNQ = \frac{A}{360} \cdot 4\pi r^2 = \frac{A}{180} \cdot 2\pi r^2.$$

17—3

If $r=1$, then will the area of the lune $APNQ$

$$=\frac{A}{180}\cdot 2\pi=\frac{\pi}{180}\cdot 2A=2\left(\frac{A^{\circ}}{r^{0}}\right)$$

= twice *the circular measure* of its angle.

67. Cor. The area of a lune, may also be expressed in terms of the corresponding arc of a small circle of the sphere, whose poles are its angular points.

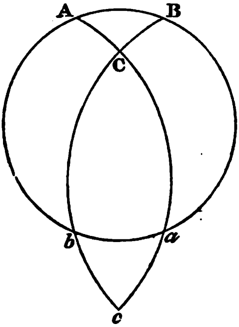
For, since $A^{0}=\angle\, AON=\angle\, aon$, we have

$$\frac{A^{0}}{r^{0}}=\frac{AN}{AO}=\frac{an}{ao}=\frac{an}{\sin Pa}:$$

and $\therefore$ the area of the lune $APNQ=\dfrac{2an}{\sin Pa}$.

68. *To express the Area of a spherical triangle, in terms of its angles.*

Let ABC be a spherical triangle on the surface of



a sphere whose radius is r: produce the sides AC, BC till they meet in c on the opposite hemisphere: then it is manifest that the arcs Cac, Cbc are semicircles: whence, since ACa, BCb are also semicircles, it follows that $AC=ac$ and $BC=bc$: therefore the angles at C, c which measure the inclination of the planes being also equal, we have the triangles ACB, acb in every respect equal to one another.

Now, if the area of the triangle $ABC=\Sigma$, and the areas BCa, ACb, Cba be called P, Q, R respectively, we shall have from Article (66):

$$\Sigma + P = \frac{A}{180}\,2\pi r^2,\ \ \Sigma + Q = \frac{B}{180}\,2\pi r^2,\ \ \Sigma + R = \frac{C}{180}\,2\pi r^2:$$

and observing that $\Sigma + P + Q + R$ = the surface of the hemisphere $= 2\pi r^2$, we obtain by addition,

$$2\Sigma + 2\pi r^2 = \frac{A+B+C}{180}\,2\pi r^2:$$

$$\text{whence, } \Sigma = \left(\frac{A+B+C-180}{180}\right)\pi r^2:$$

which is therefore *proportional* to $A + B + C - 180^\circ$, for different triangles, when the radius of the sphere is *given*.

69. Def. It has been seen in Article (21), that the sum of the angles of a spherical triangle is always greater than two right angles: and the quantity

$$A + B + C - 180^\circ$$

is therefore called the *Spherical Excess*, and usually denoted by the letter E.

70. Cor. 1. Hence the area of a triangle varies as the Spherical Excess: and if the radius of the sphere be the *lineal unit*, we have

$$\Sigma = \frac{A+B+C-180}{180^\circ}\,\pi = \frac{A+B+C-180^\circ}{r^0}$$

= *the circular measure* of the spherical excess: and it is in this sense only, that the area of a spherical triangle ABC is *numerically* equal to $A + B + C - 180^\circ$.

71. Cor. 2. It follows therefore from Article (21), that the area of a spherical triangle may be represented by any number of degrees less than 360: and also, that if two of the angles be right angles, the area varies as the remaining one.

72. Cor. 3. Since $\frac{\pi}{180^0} = \frac{\text{arc } 1''}{1''} = \frac{\sin 1''}{1''}$, nearly;

we have $\Sigma = \frac{A+B+C-180^\circ}{1''}\,r^2 \sin 1''$, nearly;

that is, the numerical value of the area of the triangle, may be obtained from multiplying the number of seconds contained in $A+B+C-180^\circ$, by .00000484813 &c. r^2.

73. A surface and an angle, being *heterogeneous* magnitudes, cannot be *immediately* compared: but for the purpose of computing the *relative* magnitudes of spherical triangular surfaces, it is evident that great facility will be attained be assuming $\Sigma = A+B+C-180^\circ$, a tacit reference being always had to what was said in Article (70).

74. *To express the area of a spherical triangle, in terms of two sides and the included angle.*

Let a, b, be the sides, and C their included angle: then $\Sigma = A+B+C-180^\circ$:

$$\text{now, } \tan\tfrac{1}{2}(A+B) = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C:$$

$$\therefore\ \cot\tfrac{1}{2}\Sigma = \cot\tfrac{1}{2}(A+B+C-180^\circ)$$

$$= \cot\{\tfrac{1}{2}(A+B+C)-90^\circ\} = -\tan\tfrac{1}{2}(A+B+C)$$

$$= -\frac{\tan\frac{1}{2}(A+B)+\tan\frac{1}{2}C}{1-\tan\frac{1}{2}(A+B)\tan\frac{1}{2}C}$$

$$= \frac{\cos\frac{1}{2}(a-b)\cot\frac{1}{2}C+\cos\frac{1}{2}(a+b)\tan\frac{1}{2}C}{\cos\frac{1}{2}(a-b)-\cos\frac{1}{2}(a+b)}$$

$$= \frac{\cos\frac{1}{2}(a-b)\cos^2\frac{1}{2}C+\cos\frac{1}{2}(a+b)\sin^2\frac{1}{2}C}{2\sin\frac{1}{2}a\sin\frac{1}{2}b\sin\frac{1}{2}C\cos\frac{1}{2}C}$$

$$= \frac{\cos\frac{1}{2}a\cos\frac{1}{2}b+\sin\frac{1}{2}a\sin\frac{1}{2}b\,(\cos^2\frac{1}{2}C-\sin^2\frac{1}{2}C)}{2\sin\frac{1}{2}a\sin\frac{1}{2}b\sin\frac{1}{2}C\cos\frac{1}{2}C}$$

$$= \frac{\cos\frac{1}{2}a\cos\frac{1}{2}b+\sin\frac{1}{2}a\sin\frac{1}{2}b\cos C}{\sin\frac{1}{2}a\sin\frac{1}{2}b\sin C}$$

$$= \frac{\cot\frac{1}{2}a\cot\frac{1}{2}b+\cos C}{\sin C}.$$

If $C=90^\circ$, $\cot\frac{1}{2}\Sigma = \cot\frac{1}{2}a\cot\frac{1}{2}b$, or $\tan\frac{1}{2}\Sigma = \tan\frac{1}{2}a\tan\frac{1}{2}b$.

75. *To express the area of a spherical triangle, in terms of the three sides.*

From the last Article we have

$$\cot \tfrac{1}{2}\Sigma = \frac{\cot\tfrac{1}{2}a \cot\tfrac{1}{2}b + \cos C}{\sin C}:$$

$$\text{but, } \cot\tfrac{1}{2}a \cot\tfrac{1}{2}b = \frac{(1+\cos a)(1+\cos b)}{\sin a \sin b}$$

$$= \frac{1+\cos a+\cos b+\cos a\cos b}{\sin a \sin b}: \cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}:$$

$$\therefore \cot\tfrac{1}{2}a \cot\tfrac{1}{2}b + \cos C = \frac{1+\cos a+\cos b+\cos c}{\sin a \sin b}:$$

$$\text{also, } \sin C = \frac{2}{\sin a \sin b}\sqrt{\sin s \sin(s-a)\sin(s-b)\sin(s-c)}:$$

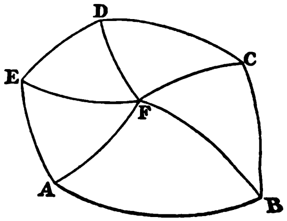
$$\therefore \cot\tfrac{1}{2}\Sigma = \frac{1+\cos a+\cos b+\cos c}{2\sqrt{\sin s \sin(s-a)\sin(s-b)\sin(s-c)}}.$$

This result, which is due to *De Gua*, is not adapted to logarithmic computation: but various other formulæ which are so adapted, have been invented: the processes, by which they are arrived at however, being laborious, though involving no difficulty, are deferred at present, and may be seen in the Second Appendix, under the head of *Theorems connected with the Spherical Excess.*

76. The area of a spherical Polygon may likewise be expressed in terms of its angles.

For, let $ABCD$ &c. be a polygon of n sides, whose angles are A, B, C, D, &c.: take any point F in its surface, from which to the angular points draw arcs of great circles of the sphere: then, the area of the polygon

$ABCD$ &c. = the sum of the areas of the triangles



AFB, BFC, CFD, DFE, &c. = the sum of the angles of the polygon + the angles at $F - n\ 180^\circ$

$$= A + B + C + D + \&c. - (n-2)\,180^\circ,$$

the nature of the equality being the same as that of the one in Article (70).

II. *The Comparison of Solid Angles.*

77. **Def.** A solid angle will evidently have the same relation to its corresponding spherical surface, whose centre is the angular point, as plane angles have to their corresponding circular arcs: and thence it follows that the magnitudes of solid angles may be expressed and compared by means of the areas of the spherical surfaces by which they are subtended.

Thus, if the solid angle be formed by three plane faces inclined to each other at the angles A, B, C: the magnitude of this angle will be proportional to, and may be expressed by, $A + B + C - 180^\circ$: and whenever a solid angle is constituted by n plane faces with the *dihedral* angles A, B, C, D, &c., its measure may be assumed to be $A + B + C + D + \&c. - (n-2)\,180^\circ$, as appears from the last article.

The *maximum limit* of *solid angular space* will manifestly be a *hemisphere,* and its *measure* the *surface* of the *hemisphere*, in the same manner as the *maximum limit* of *plane angular space, geometrically* considered, is a *semicircle*, and its *measure* the *arc* of the *semicircle.*

On this scale, therefore, the measure of the maximum solid angle will be 2π or 360°: and the magnitude of any proposed solid angle may be numerically estimated by comparing its measure with either of these quantities.

Ex. 1. In a cube, each of the solid angles is formed by three plane angles at right angles to each other: and thence we have

$$\frac{\text{the solid angle of a cube}}{\text{the maximum solid angle}} = \frac{270^\circ - 180^\circ}{360^\circ} = \frac{1}{4};$$

which we know to be true from the circumstance, that *if four* cubes be placed together upon a plane, they *exactly* fill up the angular space about a common point.

Ex. 2. In a regular right prism with a triangular base, two of the plane angles which form each solid angle are at right angles to the third, and inclined at 60^0 to each other:

$$\therefore \frac{\text{the solid angle of this prism}}{\text{the maximum solid angle}} = \frac{240^0 - 180^0}{360^0} = \frac{1}{6}.$$

$$\text{Hence, } \frac{\text{the solid angle of a cube}}{\text{the solid angle of this prism}} = \frac{1}{4} \div \frac{1}{6} = \frac{3}{2}.$$

Ex. 3. If there be two regular right prisms of m and n sides, each of the angles of their bases will be

$$\frac{m-2}{m}\, 180^0, \text{ and } \frac{n-2}{n}\, 180^0 \text{ respectively: and}$$

$$\therefore \frac{\text{the solid angle of the first prism}}{\text{the solid angle of the second prism}} = \frac{(m-2)\,n}{(n-2)\,m}.$$

Ex. 4. Let there be two regular pyramids of m and n sides, having the inclinations of two contiguous faces to each other, represented by α and β respectively: then,

$$\frac{\text{the vertical angle of the first pyramid}}{\text{the vertical angle of the second pyramid}}$$

$$= \frac{m\alpha - (m-2)\,\pi}{n\beta - (n-2)\,\pi} = \frac{2\pi - m\,(\pi - \alpha)}{2\pi - n\,(\pi - \beta)}.$$

78. Cor. 1. The same principles enable us to compare the vertical angles of right cones, by comparing the areas of the corresponding segments of spheres of equal radii, whose centres are the angular points.

Hence, since (by Article (26) of the *Differential Calculus*), the surface of a spherical segment, of given radius, varies as its altitude: if H and K be the altitudes of two cones corresponding to L the common length of their sides, we shall have

$$\frac{\text{the vertical angle of the first cone}}{\text{the vertical angle of the second cone}}$$

$$= \frac{\text{the height of the first segment}}{\text{the height of the second segment}} = \frac{L-H}{L-K} = \frac{1-\cos\theta}{1-\cos\phi},$$

if θ and ϕ be the angles of their generating triangles.

Ex. For the equilateral and right-angled cones, we have $\theta = 30^\circ$, and $\phi = 45^\circ$:

$$\therefore \frac{\text{the vertical angle of the equilateral cone}}{\text{the vertical angle of the right-angled cone}} = \frac{2-\sqrt{3}}{2-\sqrt{2}}.$$

79. Cor. 2. Whenever the spherical surface, by which any solid angle is subtended, can be divided into n parts, either equal to each other, or having any assigned ratios, the solid angle itself can be divided into n parts, having to each other the same ratios.

This method of estimating and comparing solid angles, by means of the *positions* and not the *magnitudes*, of their plane angles, was first suggested by *Albert Girard*, an ingenious *Flemish* Mathematician, in his *Invention Nouvelle en Algebre*, published about the year 1629: and it has been extended and exemplified by several modern writers.

CHAPTER V.

THE REGULAR POLYHEDRONS, THE PARALLELOPIPED AND TRIANGULAR PYRAMID.

80. *In every Polyhedron, the numbers of solid gles and plane faces together exceed the number of ges by 2.*

Let S be the number of solid angles or summits, the number of plane faces, and E the number of ges: then since every edge is common to two plane ces, $2E$ will be the number of sides of all the faces.

Within the polyhedron suppose a point to be taken m which to all the angular points straight lines may drawn: and with this point as a centre let a spherical rface be described cutting these lines, and let the ints of intersections be joined by arcs of great circles as to form as many spherical polygons as the solid has ces: let $ABCD$ &c. be one of such polygons whose mber of sides is n: then, its area

$= A + B + C + D + \&c. - (n-2)\,180^\circ$, by Article (76):

d the same being found for all the polygons, we shall ve the whole surface of the sphere or 720° equal to the m of all the angles of the polygons $-(2E-2F)\,180^\circ$:

$\therefore$ the sum of all the angles of all the polygons

$$= 720^\circ + (E-F)\,360^\circ:$$

it the sum of all the angles about any point as A ing 360°, the sum of all the angles of all the polygons ill be $S\,.\,360^\circ$:

$$\text{whence, } S\,.\,360^\circ = 720^\circ + (E-F)\,360^\circ:$$

$$\therefore\ S = 2 + E - F, \text{ or } S + F = E + 2.$$

81. Cor. Hence the sum of all the plane angles rming all the solid angles of any polyhedron

$$= (E-F)\,360^\circ = (S-2)\,360^\circ.$$

82. *To find expressions for the numbers of solid gles, plane faces, and edges of a regular polyhedron.*

Let m = the number of plane angles forming each summit:
n = the number of sides of each plane face:
then, mS = the whole number of plane angles
= the whole number of sides
= $2E$, since each edge constitutes two sides:
and nF = the whole number of sides
= $2E$:

whence, by substituting in $S+F=E+2$, we obtain

$$S=\frac{4n}{2(m+n)-mn},\ F=\frac{4m}{2(m+n)-mn},\text{ and } E=\frac{2mn}{2(m+n)-mn}.$$

83. *There can be only five regular polyhedrons.*

Since $2(m+n)-mn$ must be positive, we shall have

$$2(m+n)>mn, \text{ and } \therefore \frac{1}{m}+\frac{1}{n}>\frac{1}{2}.$$

(1) The least values of m and n being 3, and $\frac{1}{3}+\frac{1}{3}$ being $>\frac{1}{2}$, we have $S=4$, $F=4$ and $E=6$, and the solid is therefore called a *Tetrahedron.*

(2) When $m=3$ and $n=4$, since $\frac{1}{3}+\frac{1}{4}>\frac{1}{2}$, we have $S=8$, $F=6$ and $E=12$, and the solid is styled a *Hexahedron,* which is the same as a cube.

(3) When $m=3$ and $n=5$, since $\frac{1}{3}+\frac{1}{5}>\frac{1}{2}$, we have $S=20$, $F=12$ and $E=30$, and the solid is called a *Dodecahedron.*

If n be taken greater than 5, with the same value of m, the inequality $\frac{1}{m}+\frac{1}{n}>\frac{1}{2}$ will not be satisfied.

(4) When $m=4$ and $n=3$, since $\frac{1}{4}+\frac{1}{3}>\frac{1}{2}$, we have $S=6$, $F=8$ and $E=12$, and hence the solid is termed an *Octahedron.*

If values greater than 3 be assigned to n, the condition $\frac{1}{m}+\frac{1}{n}>\frac{1}{2}$ will not be fulfilled.

(5) When $m=5$ and $n=3$, since $\frac{1}{5}+\frac{1}{3}>\frac{1}{2}$, we have $S=12$, $F=20$ and $E=30$, and the solid is therefore styled an *Icosahedron.*

If m be equal to, or greater than 5, and n be greater han 3, the inequality will not hold good: and hence it ppears that there can exist *five* and *only* five regular olyhedrons.

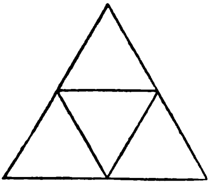
By supposing $2(m+n)-mn=0$, the values of S, F nd E become infinite, which point out the sphere as eing *ultimately* a regular polyhedron.

From this it follows that only *three* kinds of regular igures can be made to occupy the entire surface of a phere.

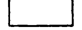
84. These are called the *Platonic Bodies*, and very orrect and clear ideas of their forms may be acquired by he following means.

Draw the diagrams underneath accurately upon *paste-oard* and cut them *out:* then if the lines in the figures be ut *half* through, so that the parts may be turned up and eet, the five regular bodies will be correctly exhibited.

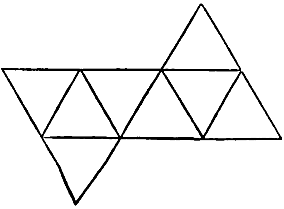
1. Tetrahedron.



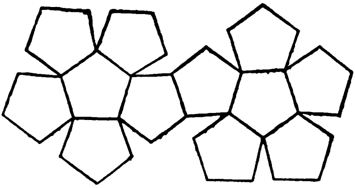
2. Hexahedron.



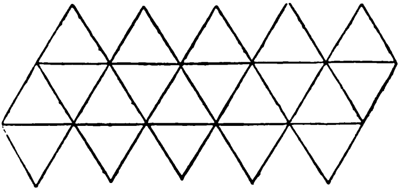
3. Octahedron.



4. Dodecahedron.

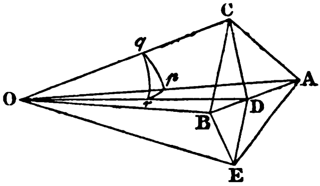


5. Icosahedron.



85. *To find the Inclination of two contiguous faces o a regular polyhedron to each other.*

Let AB be the side common to the two contiguou



faces, C and E being their centres, from which let C and ED be drawn perpendicular to it: then the ang contained between CD and ED will be the inclination these two faces to each other: in the plane in which C and ED lie, let CO and EO be drawn at right angles t them, and meeting in O: join OA, OB, OD, and fro centre O suppose a spherical surface to be describe cutting OA, OC, OD in p, q, r: then is pqr a spheric triangle, having prq a right angle:

$$\text{also, } \angle\, qpr = \tfrac{1}{2} \text{ of } \frac{360^\circ}{m} = \frac{180^\circ}{m},$$

$$\text{and } \angle\, pqr = \tfrac{1}{2} \text{ of } \frac{360^\circ}{n} = \frac{180^\circ}{n}:$$

but by *Napier's* Rules, $\cos qpr = \sin pqr \cos qr$,

$$\text{which gives } \cos qr = \cos\frac{180^\circ}{m} \div \sin\frac{180^\circ}{n}:$$

$$\text{and } \cos qr = \cos COD = \sin CDO = \sin \tfrac{1}{2} CDE:$$

$$\therefore \sin\tfrac{1}{2}CDE \text{ or } \sin\tfrac{1}{2}I = \cos\frac{180^\circ}{m} \div \sin\frac{180^\circ}{n}.$$

1) In the *Tetrahedron*, $\sin\frac{1}{2}I = \frac{1}{\sqrt{3}}$, and $\therefore \cos I = \frac{1}{3}$.

2) In the *Hexahedron*, $\sin\frac{1}{2}I = \frac{1}{\sqrt{2}}$, and $\therefore I = 90^\circ$.

3) In the *Octahedron*, $\sin\frac{1}{2}I = \sqrt{\frac{2}{3}}$, and $\therefore \cos I = -\frac{1}{3}$.

4) In the *Dodecahedron*, $\sin\frac{1}{2}I = \frac{2}{\sqrt{10 - 2\sqrt{5}}}$,

and $\therefore \cos I = -\frac{1}{5}\sqrt{5}$.

5) In the *Icosahedron*, $\sin\frac{1}{2}I = \frac{1 + \sqrt{5}}{2\sqrt{3}}$,

and $\therefore \cos I = -\frac{1}{3}\sqrt{5}$.

86. *To find the Radii of the Spheres inscribed in and ircumscribed about a regular polyhedron.*

In the last figure, CO and EO are manifestly at right ngles to the planes ABC, ABE, and equal to each other: nd the same being true for any other two contiguous aces, it follows that O is the centre of the inscribed and ircumscribed spheres whose radii are OC and OA:

$$\text{now, } \frac{OC}{OA} = \cos AOC = \cos pq = \cot qpr \cot pqr$$

$$= \cot\frac{180^\circ}{m} \cot\frac{180^\circ}{n}, \text{ by } \textit{Napier's} \text{ Rules;}$$

ut if a side of one of the faces $= a$, $CA = \frac{1}{2}a \operatorname{cosec}\frac{180^\circ}{n}$:

$$\therefore OA^2 = OC^2 + CA^2 = OC^2 + \tfrac{1}{4}a^2 \operatorname{cosec}^2\frac{180^\circ}{n}:$$

whence, if r and R denote the required radii,

we have $\frac{r}{R} = \cot\frac{180^\circ}{m}\cot\frac{180^\circ}{n}$, and $R^2 - r^2 = \frac{1}{4}a^2\operatorname{cosec}^2\frac{180^\circ}{n}$, for finding them.

87. COR. Hence $r = \frac{1}{2}a\tan\frac{1}{2}I\cot\frac{180^\circ}{n}$,

$$\text{and } R = \tfrac{1}{2}a\tan\tfrac{1}{2}I\tan\frac{180^\circ}{m}:$$

and there is now no difficulty in determining the values of r and R for the different cases.

88. *To find the Surface and Volume of a regular polyhedron.*

From the centre O of the inscribed sphere, let straight lines be drawn to all the angular points, then the polyhedron will be divided into as many pyramids with equilateral bases as there are plane faces, and whose altitudes are each equal to r: but the area of each plane face

$$= \tfrac{1}{4}na^2\cot\frac{180^\circ}{n};$$

$$\text{whence the whole surface} = \tfrac{1}{4}na^2\cot\frac{180^\circ}{n}F:$$

and the volume $= \frac{1}{3}$ of the whole Surface × the Altitude

$$= \frac{1}{12}na^2r\cot\frac{180^\circ}{n}F.$$

89. *Give the three edges of a Parallelopiped which meet, and the angles included between them, to find its Volume.*

In the same diagram, let OA, OB, OC, three edges of the parallelopiped meeting at the same angle, be denoted by a, b, c: draw CD perpendicular to the plane AOB, join OD, and let the angles AOB, AOC, BOC be called α, β, γ respectively:

$$\text{then } CD = OC\sin COD:$$

but $\sin COD = \sin qr = \sin pq\sin qpr$, by *Napier's* Rules,

$$= \sin\beta\frac{2}{\sin\alpha\sin\beta}\sqrt{\sin S\sin(S-\alpha)\sin(S-\beta)\sin(S-\gamma)}:$$

whence, the volume = the area of the base × the altitude

$$= OA\,.\,OB \sin AOB\,.\,CD$$

$$= 2abc\sqrt{\sin S \sin (S-\alpha) \sin (S-\beta) \sin (S-\gamma)}.$$

The whole surface of the parallelopiped is manifestly

$$= 2\,\{ab \sin \alpha + ac \sin \beta + bc \sin \gamma\}.$$

90. *To find the Diagonal of the Parallelopiped.*

Let $2OD$ (which, if the figure were completed, would be the diagonal of one of the faces) $= d$: then

$$d^2 = a^2 + b^2 + 2ab \cos \alpha :$$

nd if D be the diagonal required, we shall have

$$D^2 = c^2 + d^2 + 2cd \cos COD$$

$$= a^2 + b^2 + c^2 + 2ab \cos \alpha + 2cd \cos COD :$$

$$\text{but } \cos COD = \cos qr$$

$$= \cos pq \cos pr + \sin pq \sin pr \cos qpr$$

$$= \cos \beta \cos pr + \sin \beta \sin pr \left\{\frac{\cos \gamma - \cos \alpha \cos \beta}{\sin \alpha \sin \beta}\right\}$$

$$= \cos \beta \cos pr + \sin pr \left\{\frac{\cos \gamma - \cos \alpha \cos \beta}{\sin \alpha}\right\}$$

$$= \frac{\cos \gamma \sin pr}{\sin \alpha} + \frac{\cos \beta\,(\sin \alpha \cos pr - \cos \alpha \sin pr)}{\sin \alpha}$$

$$= \frac{\cos \gamma \sin pr}{\sin \alpha} + \frac{\cos \beta \sin (\alpha - pr)}{\sin \alpha}$$

$$= \cos \gamma \frac{\sin AOD}{\sin \alpha} + \cos \beta \frac{\sin BOD}{\sin \alpha} = \frac{a}{d} \cos \beta + \frac{b}{d} \cos \gamma :$$

$$\therefore D = \sqrt{a^2 + b^2 + c^2 + 2ab \cos \alpha + 2ac \cos \beta + 2bc \cos \gamma}.$$

By proper substitutions, the other diagonals are found; and thence it is easily shewn that the sum of the squares of the *four* diagonals of a parallelopiped is equal to the sum of the squares of the *twelve* edges.

91. *Given the three edges of a Triangular Pyramid which meet, and the angles they make with each other, to find its Volume.*

Since by *Euclid*, XII. 7, every prism having a triangular base may be divided into *three* equal pyramids with triangular bases, the volume of the pyramid $= \frac{1}{6}$ of that of the corresponding parallelopiped: that is,

the volume $= \frac{1}{3}\, abc \sqrt{\sin S \sin (S-\alpha) \sin (S-\beta) \sin (S-\gamma)}$.

Also, the sum of the areas of the three sides of the pyramid $= \frac{1}{2}\,\{ab \sin \alpha + ac \sin \beta + bc \sin \gamma\}$.

92. *Given the six edges of a Triangular Pyramid, to find its Volume.*

In addition to the notation used above, let the sides *AB, AC, BC* of the base be denoted by ***h, k, l***: then

$$\cos \alpha = \frac{a^2 + b^2 - h^2}{2ab}, \quad \cos \beta = \frac{a^2 + c^2 - k^2}{2ac}, \quad \cos \gamma = \frac{b^2 + c^2 - l^2}{2bc}:$$

whence, by substitution we shall have the volume

$$= \frac{1}{6}\, abc \sqrt{1 - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma}$$

$$= \frac{1}{12} \sqrt{4a^2b^2c^2 - c^2H^2 - b^2K^2 - a^2L^2 + HKL},$$

where H, K, L are the numerators in the values of $\cos \alpha$, $\cos \beta$, $\cos \gamma$ respectively.

For the Geometrical constructions of the regular polyhedrons, as well as much information upon polyhedrons in general, the student is referred to *Brewster's* Edition of *Legendre's* Geometry.

APPENDIX I.

ON PLANE TRIGONOMETRY.

CHAPTER I.

MISCELLANEOUS THEOREMS AND PROBLEMS IN PLANE TRIGONOMETRY, WITH THEIR SOLUTIONS.

It is here intended to place before the Student, the Solutions of a few Theorems and Problems, which could not, consistently with the plan of the work, be introduced into the preceding chapters, but still possess too much interest and importance to be entirely passed over in an Elementary Treatise like the present. Though some of them are of little or no practical utility, they all depend upon the principles laid down in the preceding pages, and will therefore serve to impress those principles upon the mind, and to confirm them in the memory, of the young mathematician. In order to confine the extent of the work within moderate limits, all the more minute details of Construction and Demonstration, when obvious, are omitted, the student being expected to supply them himself: and it is believed that this circumstance will have a tendency at the same time to stimulate the invention, to exercise the judgment, and to improve the taste.

I. *Trigonometrical Formulæ.*

1. Since $2\sin^2 A = 1 - \cos 2A$,

and $2\cos^2 A = 1 + \cos 2A$,

we have

$$\sin A = \pm\sqrt{\frac{1-\cos 2A}{2}}, \text{ and } \cos A = \pm\sqrt{\frac{1+\cos 2A}{2}}:$$

each admitting a *positive* and a *negative* value: but

$$\cos 2A = \cos(-2A) = \cos(360^\circ \pm 2A) = \&c.:$$

and therefore the sines and cosines of *half* these angles will some of them have one algebraical sign, and some the other, as exhibited in the formulæ above.

Similarly, for $\tan A = \pm\sqrt{\dfrac{1-\cos 2A}{1+\cos 2A}}$.

2. To consider the results obtained from the equations of Article (48): viz.

$(\cos A + \sin A)^2 = 1 + \sin 2A$, $(\cos A - \sin A)^2 = 1 - \sin 2A$.

(1) Let $2A$ be $> 0^\circ$ and $< 90^\circ$; $\therefore A > 0^\circ$ and $< 45^\circ$:

$\therefore \cos A > \sin A$, and both are positive:

whence, $\cos A + \sin A = \sqrt{1+s}$, and $\cos A - \sin A = \sqrt{1-s}$, if s be the *numerical* value of $\sin 2A$.

(2) Let $2A$ be $> 90^\circ$ and $< 180^\circ$; $\therefore A > 45^\circ$ and $< 90^\circ$:

$\therefore \cos A < \sin A$, and both are positive:

whence, $\cos A + \sin A = \sqrt{1+s}$, and $\cos A - \sin A = -\sqrt{1-s}$.

(3) Let $2A$ be $> 180^\circ$ and $< 270^\circ$; $\therefore A > 90^\circ$ and $< 135^\circ$:

$\therefore \cos A$ is negative, and $\sin A$ is positive and $> \cos A$:

also, $s = \sin 2A$, now becomes negative:

whence, $\cos A + \sin A = \sqrt{1-s}$, and $\cos A - \sin A = -\sqrt{1+s}$.

(4) Let $2A$ be $> 270^\circ$ and $< 360^\circ$; $\therefore A > 135^\circ$ and $< 180^\circ$:

$\therefore \cos A$ is negative, and $\sin A$ is positive and $< \cos A$:

whence, $\cos A + \sin A = -\sqrt{1-s}$, and $\cos A - \sin A = -\sqrt{1+s}$.

Hence, for the *first* and *second* cases respectively,

$\cos A = \frac{1}{2}\{\sqrt{1+s} \pm \sqrt{1-s}\}$: $\sin A = \frac{1}{2}\{\sqrt{1+s} \mp \sqrt{1-s}\}$:

and for the *third* and *fourth* cases respectively,

$\cos A = \frac{1}{2}\{\pm\sqrt{1-s} - \sqrt{1+s}\}$: $\sin A = \frac{1}{2}\{\pm\sqrt{1-s} + \sqrt{1+s}\}$.

That there ought to be *four* values and *no more* may be explained *a priori*, as above.

3. To express $a \cos\theta + b \cos(\theta + \alpha)$, in the form of a single function, as $A \cos(\theta + B)$.

Here, $a \cos\theta + b \cos\alpha \cos\theta - b \sin\alpha \sin\theta$

$= A \cos B \cos\theta - A \sin B \sin\theta$

$\therefore$ equating the coefficients of $\sin\theta$ and $\cos\theta$, we have

$$A\sin B = b\sin\alpha,\ \text{and}\ A\cos B = a + b\cos\alpha$$

whence, $\tan B = \dfrac{b\sin\alpha}{a + b\cos\alpha}$: and $A = \sqrt{a^2 + 2ab\cos\alpha + b^2}$.

4. To represent $\sin\alpha - \cos\beta$, as a product.

Here, $\sin\alpha - \cos\beta = \sin\alpha - \sin(90^\circ - \beta)$

$$= 2\cos\tfrac{1}{2}(\alpha + 90^\circ - \beta)\sin\tfrac{1}{2}(\alpha - 90^\circ + \beta),\ \text{by (55)},$$

$$= 2\cos\{\tfrac{1}{2}(\alpha - \beta) + 45^\circ\}\sin\{\tfrac{1}{2}(\alpha + \beta) - 45^\circ\}:$$

and similarly of other functions of this description.

5. To express in the form of a product,

$$\sin 2(\alpha - \beta) - \sin 2(\alpha - \gamma) + \sin 2(\beta - \gamma).$$

Here, taking the first two quantities, we have

$\sin 2(\alpha - \beta) - \sin 2(\alpha - \gamma) = 2\cos(2\alpha - \beta - \gamma)\sin(\gamma - \beta)$:

$\therefore$ the required product

$$= 2\cos(2\alpha - \beta - \gamma)\sin(\gamma - \beta) + \sin 2(\beta - \gamma)$$
$$= 2\sin(\beta - \gamma)\{\cos(\beta - \gamma) - \cos(2\alpha - \beta - \gamma)\}$$
$$= 2\sin(\beta - \gamma)\, 2\sin(\alpha - \gamma)\sin(\alpha - \beta)$$
$$= 4\sin(\alpha - \beta)\sin(\alpha - \gamma)\sin(\beta - \gamma).$$

6. To express by means of factors,

$$1 + \cos(2\theta - 2m\theta) - \cos(2\theta - 2\alpha) - \cos(2m\theta - 2\alpha).$$

Here, we have $\cos(2\theta - 2m\theta) - \cos(2m\theta - 2\alpha)$

$= 2\sin(\theta - \alpha)\sin(2m\theta - \alpha - \theta)$, by Article (55):

and $1 - \cos(2\theta - 2\alpha) = 2\sin^2(\theta - \alpha)$, by Article (44):

$\therefore$ the proposed quantity

$$= 2\sin^2(\theta - \alpha) + 2\sin(\theta - \alpha)\sin(2m\theta - \alpha - \theta)$$
$$= 2\sin(\theta - \alpha)\{\sin(\theta - \alpha) + \sin(2m\theta - \alpha - \theta)\}$$
$$= 2\sin(\theta - \alpha)\, 2\sin(m\theta - \alpha)\cos(m\theta - \theta)$$
$$= 4\sin(\theta - \alpha)\sin(m\theta - \alpha)\cos(m\theta - \theta).$$

This method is applicable in all similar cases.

7. To express $\sin m\theta \cos n\theta$ and $\cos m\theta \sin n\theta$, by means of simple functions.

Here, $\sin (m+n)\,\theta = \sin m\theta \cos n\theta + \cos m\theta \sin n\theta$:

$$\sin (m-n)\,\theta = \sin m\theta \cos n\theta - \cos m\theta \sin n\theta :$$

$$\therefore\ \sin m\theta \cos n\theta = \tfrac{1}{2}\{\sin (m+n)\,\theta + \sin (m-n)\theta\} :$$

$$\text{and } \cos m\theta \sin n\theta = \tfrac{1}{2}\{\sin (m+n)\,\theta - \sin (m-n)\theta\}.$$

The same process may be adopted for $\cos m\theta \cos n\theta$ and $\sin m\theta \sin n\theta$: and it is easily extended to

$$\sin m\theta \cos n\theta \sin p\theta, \ \&\text{c.}$$

Hence, we have $\sin (\alpha+\beta) \cos \beta - \sin (\alpha+\gamma) \cos \gamma$

$$= \sin (\beta-\gamma) \cos (\alpha+\beta+\gamma).$$

8. If $\alpha+\beta+\gamma = 180^0$ we have $\sin \alpha = \sin (\beta+\gamma)$:

$$\therefore\ \sin \alpha + \sin \beta + \sin \gamma = \sin (\beta+\gamma) + \sin \beta + \sin \gamma$$
$$= 2 \sin \tfrac{1}{2}(\beta+\gamma) \cos \tfrac{1}{2}(\beta+\gamma) + 2 \sin \tfrac{1}{2}(\beta+\gamma) \cos \tfrac{1}{2}(\beta-\gamma)$$
$$= 2 \sin \tfrac{1}{2}(\beta+\gamma) \{\cos \tfrac{1}{2}(\beta+\gamma) + \cos \tfrac{1}{2}(\beta-\gamma)\}$$
$$= 2 \cos \tfrac{1}{2}\alpha\ 2 \cos \tfrac{1}{2}\beta \cos \tfrac{1}{2}\gamma = 4 \cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta \cos \tfrac{1}{2}\gamma.$$

Again, $\cos^2 \alpha = (\cos \beta \cos \gamma - \sin \beta \sin \gamma)^2$

$$= \cos^2 \beta \cos^2 \gamma - 2 \sin \beta \cos \beta \sin \gamma \cos \gamma + \sin^2 \beta \sin^2 \gamma$$
$$= \cos^2 \beta \cos^2 \gamma + (1-\cos^2 \beta)(1-\cos^2 \gamma) - 2 \sin \beta \cos \beta \sin \gamma \cos \gamma$$
$$= 2 \cos^2 \beta \cos^2 \gamma + 1 - \cos^2 \beta - \cos^2 \gamma - 2 \sin \beta \cos \beta \sin \gamma \cos \gamma :$$
$$\therefore\ \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 + 2 \cos \beta \cos \gamma (\cos \beta \cos \gamma - \sin \beta \sin \gamma)$$
$$= 1 + 2 \cos \beta \cos \gamma \cos (\beta+\gamma) = 1 - 2 \cos \alpha \cos \beta \cos \gamma :$$
$$\text{or, } \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma = 1.$$

Also, if $\tan \frac{1}{2}\alpha$, $\tan \frac{1}{2}\beta$ and $\tan \frac{1}{2}\gamma$ be in arithmetical proportion, we have

$$\frac{\sin \frac{1}{2}\alpha}{\cos \frac{1}{2}\alpha} - \frac{\sin \frac{1}{2}\beta}{\cos \frac{1}{2}\beta} = \frac{\sin \frac{1}{2}\beta}{\cos \frac{1}{2}\beta} - \frac{\sin \frac{1}{2}\gamma}{\cos \frac{1}{2}\gamma},$$

$$\text{or, } \frac{\sin \frac{1}{2}(\alpha-\beta)}{\cos \frac{1}{2}\alpha} = \frac{\sin \frac{1}{2}(\beta-\gamma)}{\cos \frac{1}{2}\gamma} :$$

$$\text{but } \frac{\sin \frac{1}{2}(\alpha+\beta)}{\cos \frac{1}{2}\gamma} = \frac{\sin \frac{1}{2}(\beta+\gamma)}{\cos \frac{1}{2}\alpha}, \text{ since } \alpha+\beta+\gamma = 180^0 :$$

$$\therefore\ \sin \tfrac{1}{2}(\alpha+\beta) \sin \tfrac{1}{2}(\alpha-\beta) = \sin \tfrac{1}{2}(\beta+\gamma) \sin \tfrac{1}{2}(\beta-\gamma)$$

$$\text{or, } \cos\alpha - \cos\beta = \cos\beta - \cos\gamma :$$

and therefore $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are also in arithmetical proportion.

9. By the solution of the quadratic equation,

$$\tan A = \frac{2\tan\frac{1}{2}A}{1-\tan^2\frac{1}{2}A}, \text{ we find } \tan\tfrac{1}{2}A = \frac{-1\pm\sec A}{\tan A}:$$

and using the positive value, it is easily proved that

$$\tan\phi = \frac{\sin\theta\cos\theta'}{\sin\theta'+\cos\theta} \text{ gives } \tan\tfrac{1}{2}\phi = \tan\tfrac{1}{2}\theta\tan(45^\circ-\tfrac{1}{2}\theta').$$

10. By equal multiplication $\dfrac{\cos\alpha+\sqrt{-1}\sin\alpha}{\cos\beta+\sqrt{-1}\sin\beta}$

$$= \frac{\cos\alpha+\sqrt{-1}\sin\alpha}{\cos\beta+\sqrt{-1}\sin\beta} \times \frac{\cos\beta-\sqrt{-1}\sin\beta}{\cos\beta-\sqrt{-1}\sin\beta}$$

$$= \frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta+\sqrt{-1}(\sin\alpha\cos\beta-\cos\alpha\sin\beta)}{\cos^2\beta+\sin^2\beta}$$

$$= \cos(\alpha-\beta)+\sqrt{-1}\sin(\alpha-\beta).$$

11. If $\dfrac{m\tan\beta}{\cos^2(\alpha-\beta)} = \dfrac{n\tan(\alpha-\beta)}{\cos^2\beta}$, we have directly

$m\sin 2\beta = n\sin 2(\alpha-\beta)$:

$$\text{or, } m : n :: \sin 2(\alpha-\beta) : \sin 2\beta :$$

$$\therefore \frac{m-n}{m+n} = \frac{\sin 2(\alpha-\beta)-\sin 2\beta}{\sin 2(\alpha-\beta)+\sin 2\beta} = \frac{\tan(\alpha-2\beta)}{\tan\alpha}:$$

$$\text{whence, } \tan(\alpha-2\beta) = \frac{m-n}{m+n}\tan\alpha.$$

12. $$\frac{a^4+b^4-2a^2b^2\cos 2\theta}{a^2+b^2-2ab\cos\theta} = \frac{a^4+b^4-2a^2b^2(2\cos^2\theta-1)}{a^2+b^2-2ab\cos\theta}$$

$$= \frac{(a^2+b^2)^2-(2ab\cos\theta)^2}{(a^2+b^2)-2ab\cos\theta} = a^2+2ab\cos\theta+b^2.$$

II. *Trigonometrical Equations and Elimination.*

13. In Trigonometrical Equations, the given quantities are usually denoted by the letters α, β, &c. of the

Greek alphabet, and those which are *required*, by the letters θ, ϕ, &c., and sometimes by x, y, &c.: and the functions of these symbols are always understood to be subject to the same treatment, as any other quantities similarly concerned in an ordinary algebraical equation.

Ex. 1. If

$$\sin(\theta+\alpha)-\cos(\theta-\alpha)=\cos(\theta+\alpha)-\sin(\theta-\alpha),$$

find the values of θ.

Here, we have

$$2\cos\alpha\sin\theta=2\cos\alpha\cos\theta,\text{ or }\sin\theta=\cos\theta:$$

$$\therefore\ \tan\theta=1=\tan 45^\circ,\text{ and }\theta=45^\circ.$$

Also, $\tan\theta=\pm\tan(2r\,180^\circ\pm\theta)$

$$\text{gives }\theta=\pm(2r\,180^\circ\pm45^\circ):$$

and, from $\tan\theta=\pm\tan\{(2r+1)\,180^\circ\pm\theta\}$, we find

$$\theta=\pm\{(2r+1)\,180^\circ\pm45^\circ\}.$$

Ex. 2. Find the values of θ in the equation

$$\frac{1-\tan\theta}{1+\tan\theta}=2\cos2\theta.$$

Here, $2\cos2\theta=\dfrac{1-\tan\theta}{1+\tan\theta}=\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=\dfrac{\cos2\theta}{1+\sin2\theta}$:

$$\therefore\ 1+\sin2\theta=\tfrac{1}{2},\text{ and }\sin2\theta=-\tfrac{1}{2}=\sin(-30^\circ):$$

$$\therefore\ 2\theta=-30^\circ,\text{ and }\theta=-15^\circ.$$

Other solutions may be obtained, as in the last example: and it is also evident that the equation will be satisfied by $\cos2\theta=0=\cos90^\circ$, which gives $\theta=45^\circ$.

Ex. 3. Solve the equation

$$2\tan\theta+\tan(\alpha-\theta)=\tan(\beta+\theta).$$

Here, putting the equation in the form

$\tan\theta+\tan(\alpha-\theta)=\tan(\beta+\theta)-\tan\theta$, we have

$$\frac{\tan\alpha\,(1+\tan^2\theta)}{1+\tan\alpha\tan\theta}=\frac{\tan\beta\,(1+\tan^2\theta)}{1-\tan\beta\tan\theta}:$$

whence, $\tan\theta=\dfrac{\tan\alpha-\tan\beta}{2\tan\alpha\tan\beta}=\dfrac{\sin(\alpha-\beta)}{2\sin\alpha\sin\beta}$.

The artifice here used greatly facilitates the reductions: and $1 + \tan^2\theta = 0$, which also satisfies the *transformed* equation, gives $\tan\theta = \pm\sqrt{-1}$, two imaginary roots, which do not belong to the equation *proposed*.

Ex. 4. Determine the general forms of θ, which satisfy the equation, $\sin 2\theta = \cos 3\theta$.

Here, $2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$, from which we obtain $\cos\theta = 0 = \cos 90^\circ$, or $\theta = 90^\circ$:

$$\text{also, } \sin\theta = \frac{\pm\sqrt{5} - 1}{4}, \text{ and } \cos\theta = \pm\frac{\sqrt{10 \pm 2\sqrt{5}}}{4}.$$

Now, $\sin 2\theta = \pm\sin(2r\,180^\circ \pm 2\theta)$, by Article (116):

$$= \cos 3\theta = \sin(90^\circ - 3\theta):$$

(1) $2r\,180^\circ + 2\theta = 90^\circ - 3\theta$, gives

$$5\theta = 90^\circ - 2r\,180^\circ = 90^\circ(1 - 4r), \text{ and } \theta = -(4r - 1)\,18^\circ:$$

(2) $2r\,180^\circ - 2\theta = -90^\circ + 3\theta$, gives

$$5\theta = 90^\circ + 2r.180^\circ = 90^\circ(1 + 4r), \text{ and } \theta = (4r + 1)\,18^\circ.$$

Again, $\sin 2\theta = \pm\sin\{(2r + 1)\,180^\circ \mp 2\theta\}$

$$= \cos 3\theta = \sin(90^\circ - 3\theta).$$

(3) $(2r + 1)\,180^\circ - 2\theta = 90^\circ - 3\theta$, gives

$$\theta = 90^\circ - (2r + 1)\,180^\circ = -\{4r + 1\}\,90^\circ:$$

(4) $(2r + 1)\,180^\circ + 2\theta = -90^\circ + 3\theta$, gives

$$\theta = 90^\circ + (2r + 1)\,180^\circ = (4r + 3)\,90^\circ.$$

The negative values of $\sin\theta$ and $\cos\theta$ above found, belong to an angle of the form $(4r + 1)\,18^\circ$, which is *greater* than *two*, and *less* than *three* right angles.

By means of this example it is easily shewn that all the values of θ, which satisfy the equation

$$\sin^2 2\theta - \sin^2\theta = \sin^2\frac{\pi}{6},$$

are comprised in the expression $(r + \frac{1}{5} \pm \frac{1}{10})\,\pi$.

Ex. 5. If $\sec\alpha\sec\theta + \tan\alpha\tan\theta = \sec\beta$, find $\tan\theta$.

Here, $\sec\alpha\sec\theta = \sec\beta - \tan\alpha\tan\theta$; and squaring both sides of the equality, we have

$$\sec^2\alpha + \sec^2\alpha\tan^2\theta = \sec^2\beta - 2\tan\alpha\sec\beta\tan\theta + \tan^2\alpha\tan^2\theta:$$

$$\therefore\ \tan^2\theta + 2\tan\alpha\sec\beta\tan\theta = \sec^2\beta - \sec^2\alpha:$$

and completing the square, we have directly

$$(\tan\theta + \tan\alpha\sec\beta)^2 = \sec^2\beta - \sec^2\alpha + \tan^2\alpha\sec^2\beta$$

$$= \sec^2\beta(1 + \tan^2\alpha) - \sec^2\alpha = \sec^2\alpha\sec^2\beta - \sec^2\alpha$$

$$= \sec^2\alpha(\sec^2\beta - 1) = \sec^2\alpha\tan^2\beta:$$

$$\therefore\ \tan\theta + \tan\alpha\sec\beta = \pm\sec\alpha\tan\beta:$$

whence, $\tan\theta = \pm\{\sec\alpha\tan\beta \mp \tan\alpha\sec\beta\} = \pm\dfrac{\sin\beta\mp\sin\alpha}{\cos\alpha\cos\beta}$.

Ex. 6. Given $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, to find the values of c.

Here, $c^2-2bc\cos A = a^2-b^2$, and completing the square, &c. we have

$$c^2 - 2bc\cos A + b^2\cos^2 A = a^2 - b^2 + b^2\cos^2 A = a^2 - b^2\sin^2 A:$$

whence, $c = b\cos A \pm \sqrt{a^2 - b^2\sin^2 A}$.

This example is here introduced because it comprises the *ambiguous* case of the solution of triangles.

(1) If $A > 90^0$, $b\cos A$ is *negative*, and in order that c may be *positive*, the *positive* sign must be used.

(2) If $A < 90^0$ and $b < a$, we shall have

$$\sqrt{a^2 - b^2\sin^2 A} = \sqrt{a^2 - b^2 + b^2\cos^2 A}$$

greater than $b\cos A$, so that c can be *positive* only when the *positive* sign is used, or when $c = b\cos A + \sqrt{a^2 - b^2\sin^2 A}$.

(3) If $A < 90^0$ and $b > a$, then $\sqrt{a^2 - b^2\sin^2 A}$ is less than $b\cos A$, and *both* signs will render the value of c *positive:* or, the solution is *ambiguous*.

(4) If $\sin A = \dfrac{a}{b}$, or $a^2 - b^2\sin^2 A = 0$, we shall have $c = b\cos A$, and therefore the triangle is *right-angled* at B: but if $\sin A > \dfrac{a}{b}$, or $b^2\sin^2 A > a^2$, the triangle becomes *impossible*. See Article (110).

Ex. 7. Given $\sin\theta = m\sin\phi$, and $\tan\theta = n\tan\phi$: *to find the sines and cosines of* θ *and* ϕ.

Here, $\frac{\sin\theta}{\cos\theta} = n\frac{\sin\phi}{\cos\phi}$, or $\frac{\sin\theta}{\sin\phi} = n\frac{\cos\theta}{\cos\phi}$:

$$\therefore \frac{m^2}{n^2} = \frac{\cos^2\theta}{\cos^2\phi} = \frac{1-\sin^2\theta}{1-\sin^2\phi} = \frac{1-m^2\sin^2\phi}{1-\sin^2\phi}:$$

vhence, $\sin\phi = \pm\frac{1}{m}\sqrt{\frac{m^2-n^2}{1-n^2}}$, and $\sin\theta = \pm\sqrt{\frac{m^2-n^2}{1-n^2}}$:

lso, $\cos\phi = \pm\frac{n}{m}\sqrt{\frac{1-m^2}{1-n^2}}$, and $\cos\theta = \pm\sqrt{\frac{1-m^2}{1-n^2}}$.

Ex. 8. Eliminate θ by means of the equations,

$$\sec\theta - \cos\theta = m, \text{ and } \operatorname{cosec}\theta - \sin\theta = n.$$

Here, we have $m = \frac{\sin^2\theta}{\cos\theta}$, and $n = \frac{\cos^2\theta}{\sin\theta}$:

whence, $mn = \sin\theta\cos\theta$: and $\therefore m^2n = \sin^3\theta$:

also, $mn^2 = \sin\theta\cos\theta \times \frac{\cos^2\theta}{\sin\theta} = \cos^3\theta$:

and therefore, $1 = \sin^2\theta + \cos^2\theta = m^{\frac{4}{3}}n^{\frac{2}{3}} + m^{\frac{2}{3}}n^{\frac{4}{3}}$.

Ex. 9. Given $3\cos^2\theta = m^2 - 1$, and $\tan^3\frac{1}{2}\theta = \tan\alpha$, o eliminate θ.

Here, $m^2 = 1 + 3\cos^2\theta = 4 - 3\sin^2\theta = 4 - 3\left(\frac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}\right)^2$

$$= 4\left\{\frac{1-\tan^2\frac{1}{2}\theta + \tan^4\frac{1}{2}\theta}{(1+\tan^2\frac{1}{2}\theta)^2}\right\} = 4\left\{\frac{1+\tan^6\frac{1}{2}\theta}{(1+\tan^2\frac{1}{2}\theta)^3}\right\}$$

$$= 4\left\{\frac{1+\tan^2\alpha}{(1+\tan^{\frac{2}{3}}\alpha)^3}\right\} = 4\left\{\frac{\sin^2\alpha+\cos^2\alpha}{(\cos^{\frac{2}{3}}\alpha+\sin^{\frac{2}{3}}\alpha)^3}\right\} = \frac{4}{(\cos^{\frac{2}{3}}\alpha+\sin^{\frac{2}{3}}\alpha)^3}:$$

vhence, $m(\cos^{\frac{2}{3}}\alpha + \sin^{\frac{2}{3}}\alpha)^{\frac{3}{2}} = \pm 2$, is the resulting equation.

Ex. 10. Eliminate θ and ϕ from the equations:

$$a\sin^2\theta + b\cos^2\theta = c,\quad b\sin^2\phi + a\cos^2\phi = d,$$
$$\text{and } a\tan\theta = b\tan\phi.$$

From (1), $a\sin^2\theta + b\cos^2\theta = c$:

also, $\underline{a\sin^2\theta + a\cos^2\theta = a}$:

$$\therefore (a-b)\cos^2\theta = a - c:$$

$$\text{from (2)},\quad b\sin^2\phi + a\cos^2\phi = d:$$

$$\text{also},\quad \underline{b\sin^2\phi + b\cos^2\phi = b}:$$

$$\therefore\ (a-b)\cos^2\phi = d-b;$$

$$\therefore\ \cos^2\theta = \frac{a-c}{a-b},\ \text{and}\ \sin^2\theta = 1 - \frac{a-c}{a-b} = \frac{c-b}{a-b}:$$

$$\cos^2\phi = \frac{d-b}{a-b},\ \text{and}\ \sin^2\phi = 1 - \frac{d-b}{a-b} = \frac{a-d}{a-b}:$$

and substituting these values in the third equation,

$$\frac{a^2\sin^2\theta}{\cos^2\theta} = \frac{b^2\sin^2\phi}{\cos^2\phi},\ \text{we find}\ \frac{a^2(c-b)}{a-c} = \frac{b^2(a-d)}{d-b}:$$

$$\text{which, being reduced, gives}\ \frac{1}{a} + \frac{1}{b} = \frac{1}{c} + \frac{1}{d}.$$

III. *Inverse Trigonometrical Functions.*

14. Def. If $\sin A = s$, $\cos A = c$, $\tan A = t$, &c.: then is A an angle whose sine, cosine, tangent, &c. are s, c, t, &c. respectively: and accordingly it is termed an *inverse* angular function of these latter quantities.

Instead of writing the description of A at *length*, it is usual to express its dependence upon the quantities, s, c, t, &c. by the following *peculiar* notation:

$$A = \sin^{-1}s = \cos^{-1}c = \tan^{-1}t = \&\text{c.}:$$

wherein it is to be carefully remembered that the index -1, does not imply an algebraical *operation*, but merely expresses the connection between the angle and its functions as above enunciated.

Thus, since $\sin 30^\circ = \frac{1}{2}$, we have $30^\circ = \sin^{-1}(\frac{1}{2})$: also this gives $\sin^{-1}\sin\alpha = \alpha$, and $\sin\sin^{-1}s = s$.

15. Since $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, we have $\alpha + \beta = \sin^{-1}\{\sin\alpha\cos\beta + \cos\alpha\sin\beta\}$: thus,

$$\text{if}\ \sin\alpha = \frac{3}{5},\ \sin\beta = \frac{4}{5},\ \text{we have}\ \cos\alpha = \frac{4}{5},\ \cos\beta = \frac{3}{5};$$

$$\textit{whence},\ \alpha = \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{4}{5},\ \text{and}\ \beta = \sin^{-1}\frac{4}{5} = \cos^{-1}\frac{3}{5}:$$

$\therefore\ \sin^{-1}\frac{3}{5}+\sin^{-1}\frac{4}{5}=\sin^{-1}(1)=90^\circ$, by substitution.

Similarly, for $\sin(\alpha-\beta)$, and $\cos(\alpha\pm\beta)$.

16. $\operatorname{Tan}(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$ may be represented in the form, $\alpha+\beta=\tan^{-1}\left\{\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\right\}$:

$\therefore$ if $\tan\alpha=t_1$, or $\alpha=\tan^{-1}t_1$; $\tan\beta=t_2$, or $\beta=\tan^{-1}t_2$;

we shall have $\tan^{-1}t_1+\tan^{-1}t_2=\tan^{-1}\left\{\dfrac{t_1+t_2}{1-t_1t_2}\right\}$.

Thus, $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\left(\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)=\tan^{-1}(1)=45^\circ$.

$$\text{Similarly, }\tan^{-1}t_1-\tan^{-1}t_2=\tan^{-1}\left\{\frac{t_1-t_2}{1+t_1t_2}\right\}.$$

$$\text{Hence, }\tan^{-1}a-\tan^{-1}b=\tan^{-1}\frac{a-b}{1+ab}:$$

$$\text{and }\tan^{-1}b-\tan^{-1}c=\tan^{-1}\frac{b-c}{1+bc}:$$

$$\therefore\ \tan^{-1}a-\tan^{-1}c=\tan^{-1}\frac{a-b}{1+ab}+\tan^{-1}\frac{b-c}{1+bc}.$$

17. If $\alpha=\sin^{-1}\sqrt{\dfrac{x}{a}}$, we have $\sin\alpha=\sqrt{\dfrac{x}{a}}$:

whence, $\cos\alpha=\sqrt{\dfrac{a-x}{a}}$, $\tan\alpha=\sqrt{\dfrac{x}{a-x}}$, &c.

$\therefore\ \alpha=\sin^{-1}\sqrt{\dfrac{x}{a}}=\cos^{-1}\sqrt{\dfrac{a-x}{a}}=\tan^{-1}\sqrt{\dfrac{x}{a-x}}=$ &c.:

that is, $\sin^{-1}\sqrt{\dfrac{x}{a}}=\cos^{-1}\sqrt{\dfrac{a-x}{a}}=\tan^{-1}\sqrt{\dfrac{x}{a-x}}=$ &c.

18. By substitution and reduction, we easily find

$$\tan^{-1}\left(\frac{x\cos\phi}{1-x\sin\phi}\right)-\tan^{-1}\left(\frac{x-\sin\phi}{\cos\phi}\right)=\phi.$$

19. Find x in $\tan^{-1}\dfrac{1}{x-1}-\tan^{-1}\dfrac{1}{x+1}=15°$.

Here, we have $\tan^{-1}\dfrac{2}{x^2}=\tan^{-1}\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$; and suppressing $\tan^{-1}$ in both sides, we find $x=\pm(\sqrt{3}+1)$.

20. From $\tan 2a=\dfrac{2\tan a}{1-\tan^2 a}$, if $a=\tan^{-1}t$, we obtain

$$2\tan^{-1}t=\tan^{-1}\frac{2t}{1-t^2}:$$

also, from $\sin 2a=2\sin a\cos a$, if $a=\sin^{-1}s=\cos^{-1}\sqrt{1-s^2}$, we have $2\sin^{-1}s=\sin^{-1}(2s\sqrt{1-s^2})$: &c.

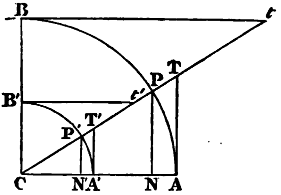
Hence, in the equation $2\tan^{-1}t=\sin^{-1}2s$, we may find the relation between s and t:

$$\text{thus, } \tan^{-1}\frac{2t}{1-t^2}=\sin^{-1}2s=\tan^{-1}\frac{2s}{\sqrt{1-4s^2}}:$$

$$\therefore \frac{t}{1-t^2}=\frac{s}{\sqrt{1-4s^2}} \text{ gives immediately } s=\pm\frac{t}{1+t^2}.$$

21. To establish and explain the relation expressed by the equation, $\sin^{-1}s=r\sin^{-1}\dfrac{s}{r}$.

Let the angle ACP or $A'CP'$, be measured by the



arcs AP, $A'P'$, whose radii CA, CA' are r and 1 respectively: then we have

$$\frac{N'P'}{NP}=\frac{CP'}{CP}=\frac{1}{r}, \text{ or } N'P'=\frac{NP}{r}:$$

$$\text{also, } \frac{\text{arc } AP}{\text{arc } A'P'}=\frac{CA}{CA'}=r, \text{ or arc } AP=r \text{ arc } A'P':$$

whence, expressing this equality by means of the notation above explained, where NP is denoted by s, we have

$$\sin^{-1} s = r \sin^{-1} N'P' = r \sin^{-1} \frac{s}{r}.$$

Similarly, $\cos^{-1} c = r \cos^{-1} \frac{c}{r}$, $\text{vers}^{-1} v = r \text{ vers}^{-1} \frac{v}{r}$, &c.

From this it appears, that the equalities just considered, are admissible only when the *circular* functions are employed, because the *angle* is the same for both their members.

22. Since, $\tan^{-1} t_1 - \tan^{-1} t_2 = \tan^{-1} \dfrac{t_1 - t_2}{1 + t_1 t_2}$:

$$\tan^{-1} t_2 - \tan^{-1} t_3 = \tan^{-1} \frac{t_2 - t_3}{1 + t_2 t_3}: \text{ \&c}:$$

$$\tan^{-1} t_{n-1} - \tan^{-1} t_n = \tan^{-1} \frac{t_{n-1} - t_n}{1 + t_{n-1} t_n}:$$

by addition we obtain the following curious theorem:

$$\tan^{-1} t_1 - \tan^{-1} t_n$$

$$= \tan^{-1} \frac{t_1 - t_2}{1 + t_1 t_2} + \tan^{-1} \frac{t_2 - t_3}{1 + t_2 t_3} + \text{\&c.} + \tan^{-1} \frac{t_{n-1} - t_n}{1 + t_{n-1} t_n}.$$

23. If we put the theorem in the following form:

$$\tan^{-1} t_1 = \tan^{-1} \frac{t_1 - t_2}{1 + t_1 t_2} + \tan^{-1} \frac{t_2 - t_3}{1 + t_2 t_3} + \text{\&c.}$$

$$+ \tan^{-1} \frac{t_{n-1} - t_n}{1 + t_{n-1} t_n} + \tan^{-1} t_n:$$

it will enable us to divide any given angle into as many parts as we please, whose tangents may be assigned.

Thus, if $t_1 = 1$, and $t_2 = \frac{1}{2}$, we shall have

$\dfrac{t_1 - t_2}{1 + t_1 t_2} = \frac{1}{3}$, so that $\tan^{-1} 1 = \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{2}$.

If we make $t_1 = 1$, $\dfrac{t_1 - t_2}{1 + t_1 t_2} = \frac{1}{3}$, $\dfrac{t_2 - t_3}{1 + t_2 t_3} = \frac{1}{5}$, $\dfrac{t_3 - t_4}{1 + t_3 t_4} = \frac{1}{7}$:

we shall have $t_2 = \frac{1}{2}$, $t_3 = \frac{3}{11}$, and $t_4 = \frac{1}{8}$:

$$\therefore \frac{\pi}{4} = \tan^{-1} \tfrac{1}{3} + \tan^{-1} \tfrac{1}{5} + \tan^{-1} \tfrac{1}{7} + \tan^{-1} \tfrac{1}{8}.$$

and $\frac{1}{4}\pi$ has thus been divided into *four* parts, whose ta gents are $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$ and $\frac{1}{8}$ respectively.

24. To the Student who may be desirous of fa liarity with this notation, it is recommended to ver the following results.

(1) $\text{Sin}^{-1}\frac{1}{2}+\sin^{-1}\frac{1}{4}=\sin^{-1}\frac{\sqrt{3}}{8}(\sqrt{5}+1).$

(2) $\text{Cos}^{-1}\sqrt{\frac{2}{3}}+\cos^{-1}\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}}=60^\circ.$

(3) $\text{Sin}^{-1}\frac{1}{\sqrt{3}}-\cos^{-1}\left(\frac{1}{\sqrt{2}}+\frac{1}{2\sqrt{3}}\right)=30^\circ.$

(4) $\text{Sin}^{-1}(\frac{1}{2}\sqrt{2+\sqrt{2}})+\sin^{-1}(\frac{1}{2}\sqrt{2-\sqrt{2}})=90^\circ.$

(5) $\text{Sin}^{-1}\frac{1}{\sqrt{50}}+\sin^{-1}\frac{1}{\sqrt{65}}=\sin^{-1}\left(\frac{3}{\sqrt{130}}\right).$

(6) $\text{Tan}^{-1}\frac{1}{\sqrt{3}}+\tan^{-1}\frac{1}{\sqrt{15}}=\sin^{-1}(\sin 60^\circ\cos 36^\circ).$

(7) $\text{Tan}^{-1}\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)-\tan^{-1}\frac{1}{\sqrt{2}}=45^\circ.$

(8) $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=45^\circ.$

(9) $\text{Sin}^{-1}\frac{x}{\sqrt{1+x^2}}+\sin^{-1}\frac{1}{\sqrt{1+x^2}}=90^\circ=\tan^{-1}x+\tan^{-}$

(10) $\text{Cos}^{-1}\frac{3}{4}+\cos^{-1}\frac{9}{16}+\cos^{-1}\frac{1}{8}=180^\circ.$

(11) $2\tan^{-1}\frac{1}{8}+2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}=45^\circ.$

(12) $\text{Cot}^{-1}\frac{1}{2}+\cot^{-1}\frac{1}{3}-\cot^{-1}\frac{1}{2+\sqrt{3}}=60^\circ.$

(13) $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}=\frac{1}{4}\pi.$

(14) $\text{Tan}^{-1}\frac{1}{99}-\tan^{-1}\frac{1}{70}+4\tan^{-1}\frac{1}{5}=\frac{1}{4}\pi.$

(15) $8\tan^{-1}\frac{1}{10}-4\tan^{-1}\frac{1}{515}-\tan^{-1}\frac{1}{239}=\frac{1}{4}\pi.$

It will readily be seen that, by means of *Gregory's* series established in Article (133), several of these results may afford approximations to the value of $\frac{1}{4}\pi$: and by reason of the *smallness* of the component factors, it will not be laborious to compute its value accurately to several places of decimals from the formula,

$$\tfrac{1}{4}\pi = \tan^{-1}\frac{1}{99} - \tan^{-1}\frac{1}{70} + 4\tan^{-1}\frac{1}{5}:$$

or, by reason of the *rapidity* of convergence of the two latter series, from the formula,

$$\tfrac{1}{4}\pi = 8\tan^{-1}\frac{1}{10} - 4\tan^{-1}\frac{1}{515} - \tan^{-1}\frac{1}{239}.$$

The notation employed in these articles is what was alluded to in Article (22): moreover, if the sine, cosine, &c. of an arc α_1, be supposed to be formed into an arc α_2, then $\sin\alpha_2 = \sin(\sin\alpha_1)$ is sometimes expressed by $\sin^2\alpha_1$: in the same manner, $\sin\{\sin(\sin\alpha_1)\}$ is written $\sin^3\alpha_1$: and similarly of the other functions.

IV. *Triangles and their Properties.*

25. To ascertain the limitations under which three straight lines can form a triangle.

Here the cosines of all the angles must lie *between* the limits $+1$ and -1: whence, if a denote the side which is not less than either of the others, and $a - b < c$, $a - c < b$, $b - c < a$, we shall have

$$a^2 - 2ab + b^2 < c^2,\ a^2 - 2ac + c^2 < b^2,\ b^2 - 2bc + c^2 < a^2:$$

or, $a^2 + b^2 - c^2 < 2ab$, $a^2 + c^2 - b^2 < 2ac$, $b^2 + c^2 - a^2 < 2bc$:

thus, $\cos C$, $\cos B$, $\cos A$, are each less than 1, and the triangle will be *possible*.

Also, $-2bc < a^2 - b^2 - c^2$, or $a^2 - b^2 - c^2 > -2bc$

shews, that if the cosine of A be negative, it is greater than -1, and the triangle will be *possible*.

If $a = b + c$, we shall have $a - b = c$, and $a - c = b$:

$\therefore b^2 + c^2 - a^2 = -2bc$, $a^2 + b^2 - c^2 = 2ab$, $a^2 + c^2 - b^2 = 2ac$ $\therefore$

whence, $\cos A = -1 = \cos 180^\circ$, or $A = 180^\circ$; $\cos C = 1 = \cos 0^\circ$,

$$\text{or } C = 0; \text{ and } \cos B = 1 = \cos 0^\circ, \text{ or } B = 0;$$

and the triangle merges in a *straight line.*

If $a > b + c$, then will $a - b > c$, and $a - c > b$:

$\therefore\ a^2 - b^2 - c^2 > 2bc,\ a^2 + b^2 - c^2 > 2ab,\ a^2 + c^2 - b^2 > 2ac$:

which give each of the cosines > 1, and therefore shew that the lines a, b, c *cannot form* a triangle.

Again, $\sin \frac{1}{2} A = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$, and if $a > b + c$,

$$2(s-b) = a + c - b > b + c + c - b > 2c, \text{ or } s - b > c:$$

$$2(s-c) = a + b - c > b + c + b - c > 2b, \text{ or } s - c > b:$$

whence, $\sin \frac{1}{2} A > 1$, which is *impossible.*

Also, $2(s-a) = b + c - a$ being *negative*, we shall likewise have $\sin \frac{1}{2} B$ and $\sin \frac{1}{2} C$ *imaginary.*

Lastly, $bc \sin A = 2\sqrt{s(s-a)(s-b)(s-c)}$, can be *real* only when $s-a$, $s-b$, $s-c$ are *all* positive as hitherto; or, when *two* of them as $s-b$ and $s-c$ are negative:

$$\text{but } 2(s-b) < 0 \text{ gives } a + c - b < 0;$$

$$\text{and } 2(s-c) < 0 \text{ gives } a + b - c < 0:$$

whence, by addition we have $2a < 0$; or one side of the triangle is *negative*, which is necessarily excluded.

The circumstances here pointed out, are nothing more than what have all along been assumed, that any two sides of a triangle are together greater than the third: but it thus appears that the *Algebraical Formulæ* are sufficient of themselves to determine the consistency of the data for the purpose, or otherwise.

26. Given A, B and $a \pm b$, to solve the triangle.

$$\text{Tan} \tfrac{1}{2}(A - B) = \frac{a-b}{a+b} \tan \tfrac{1}{2}(A + B) = \frac{a-b}{a+b} \cot \tfrac{1}{2} C:$$

which gives $a - b$, or $a + b$: and thus a and b are found.

27. If A, a and $b + c$ be given, we have

$$4 \cos^2 \tfrac{1}{2} A = \frac{(a+b+c)(b+c-a)}{bc},$$

from which bc is known: and thus b and c may be found

28. Given the perimeter, the area and the angle A f a triangle, to find the side a.

$$\text{Here, } 4\cos^2 \tfrac{1}{2}A = \frac{(b+c)^2 - a^2}{bc} = \frac{(2s-a)^2 - a^2}{bc},$$

$$\text{and } S = \tfrac{1}{2}\, bc \sin A :$$

$$\therefore\ a = s - \frac{S}{s} \cot \tfrac{1}{2}A \text{ becomes known.}$$

29. To find a point in the plane of a triangle, at vhich the sides shall subtend given angles.

Let ABC be the triangle, O the required point: lraw OA, OB, OC: then if $\angle\, AOC = \alpha$, $\angle\, BOC = \beta$, $OAC = \theta$, and $\angle\, OBC = \phi$, we shall have

$$\frac{\sin \alpha}{\sin \theta} = \frac{AC}{OC} = \frac{b}{OC};\quad \frac{\sin \phi}{\sin \beta} = \frac{OC}{BC} = \frac{OC}{a};$$

$$\therefore\ \frac{\sin \alpha \sin \phi}{\sin \beta \sin \theta} = \frac{b}{a}, \text{ or } a \sin \alpha \sin \phi = b \sin \beta \sin \theta :$$

ow, $\angle AOB = 360^\circ - \alpha - \beta$, $\angle OAB = A - \theta$, $\angle OBA = B - \phi$:

vhence, $180^\circ = 360^\circ + A + B - (\alpha + \beta + \theta + \phi)$;

$$\therefore\ \phi = 360^\circ - (\alpha + \beta + C + \theta) = \gamma - \theta, \text{ suppose}:$$

$$\text{and } b \sin \beta \sin \theta = a \sin \alpha \sin (\gamma - \theta)$$

$$= a \sin \alpha \sin \gamma \cos \theta - a \sin \alpha \cos \gamma \sin \theta,$$

$$\text{gives } \tan \theta = \frac{a \sin \alpha \sin \gamma}{b \sin \beta + a \sin \alpha \cos \gamma}:$$

nd thus, the lines OA, OB, OC may be determined.

If $\alpha = \beta = 120^\circ$, or the sides of the triangle subtend *qual* angles at the point O, we have $\sin \alpha = \sin \beta$:

$$\therefore\ \tan \theta = \frac{a \sin \gamma}{b + a \cos \gamma} \text{ and } \sin \theta = \frac{a \sin \gamma}{\sqrt{a^2 + 2ab \cos \gamma + b^2}}:$$

vhence, $\dfrac{CO}{CA} = \dfrac{\sin \theta}{\sin 120^\circ}$, or $CO = \dfrac{2ab \sin \gamma}{\sqrt{3(a^2 + 2ab \cos \gamma + b^2)}}$:

ut $\gamma = 120^\circ - C$, $\therefore\ CO = \dfrac{2ab \sin (60^\circ + C)}{\sqrt{3(a^2 - 2ab \cos (60^\circ + C) + b^2)}}$.

Similarly, AO and BO become known, and it appears that $\sin \theta : \sin \phi :: a : b :: \sin A : \sin B$.

This problem may be solved *geometrically*, by describing on AC, BC, two segments of circles containing angles equal to α, β, respectively, and intersecting in O.

30. When $\alpha = \beta = 120^\circ$, let $AO = p$, $BO = q$, $CO = r$:

$$\text{then, } 2S = \tfrac{1}{2}\sqrt{3}\,(pq + pr + qr), \text{ or } \frac{4S}{\sqrt{3}} = pq + pr + qr:$$

$$\text{but } a^2 = q^2 + r^2 + qr,\ b^2 = p^2 + r^2 + pr,\ c^2 = p^2 + q^2 + pq:$$

$$\therefore\ a^2 + b^2 + c^2 = 2\,(p^2 + q^2 + r^2) + \frac{4S}{\sqrt{3}}$$

$$= 2\left\{(p + q + r)^2 - 2\,(pq + pr + qr) + \frac{2S}{\sqrt{3}}\right\}$$

$$= 2(p+q+r)^2 - 4\sqrt{3}S: \text{ or, } (p+q+r)^2 = \tfrac{1}{2}(a^2+b^2+c^2) + 2\sqrt{3}S.$$

31. If CD meet the base of the triangle ABC in D; then if $\angle\, BAC = A$, $\angle\, DCB = \gamma$, $AD = h$, and $DB = k$, be given, the triangle may be solved.

For, if $\angle\, CDB = \theta$, we shall have $CD = \dfrac{h \sin A}{\sin(\theta - A)}$:

$$\text{also, } \angle\, CBD = 180 - (\gamma + \theta) : \therefore\ CD = \frac{k \sin(\gamma + \theta)}{\sin\gamma}:$$

$$\text{whence, } k \sin(\theta - A) \sin(\gamma + \theta) = h \sin A \sin\gamma,$$

$$\text{which gives } \cos(2\theta + \gamma - A) = \cos(A + \gamma) - \frac{2h}{k} \sin A \sin\gamma:$$

and thus all the parts of the triangle may be found.

If α denote the *least positive* angle, whose cosine = the latter member, the values of θ will be had from the two equalities,

$$2\theta + \gamma - A = \alpha, \text{ or } \theta = \tfrac{1}{2}(A + \alpha - \gamma):$$

$$2\theta + \gamma - A = 2\pi - \alpha, \text{ or } \theta = \tfrac{1}{2}(2\pi + A - \alpha - \gamma).$$

In cases of this kind, care must be taken to select only such values of the required angle as may consist with the nature of the problem, and the *geometrical construction* will here confirm the correctness of the choice.

32. Through any point O in the plane of a triangle ABC, let straight lines Aa, Bb, Cc be drawn to meet the *sides* in a, b, c respectively: then

$$
\begin{aligned}
&Ab : AO :: \sin AOb : \sin AbO,\\
&AO : Ac :: \sin AcO : \sin AOc:\\
\therefore\ &Ab : Ac :: \sin AOb \sin AcO : \sin AbO \sin AOc:\\
\text{so, }\ &Bc : Ba :: \sin BOc \sin BaO : \sin BcO \sin BOa:\\
&Ca : Cb :: \sin COa \sin CbO : \sin CaO \sin COb:
\end{aligned}
$$

$$\therefore\ Ab.Bc.Ca : Ac.Ba.Cb :: 1 : 1,$$

by reason of *vertical* and *supplemental* angles:

that is, $Ab \,.\, Bc \,.\, Ca = Ac \,.\, Ba \,.\, Cb.$

Again, $\dfrac{Ab}{Cb} = \dfrac{AB \sin ABO}{BC \sin CBO}$, $\dfrac{Bc}{Ac} = \dfrac{BC \sin BCO}{AC \sin ACO}$,

and $\dfrac{Ca}{Ba} = \dfrac{AC \sin CAO}{AB \sin BAO}$: whence we have

$$1 = \frac{\sin ABO \sin BCO \sin CAO}{\sin CBO \sin ACO \sin BAO}:$$

or, $\sin ABO \sin BCO \sin CAO = \sin CBO \sin ACO \sin BAO.$

33. Conversely, when either of these equalities is satisfied, it follows that the three straight lines Aa, Bb, Cc pass through the *same* point.

(1) When the sides of the triangle are *bisected* in a, b, c, the theorem evidently holds good.

(2) When Aa, Bb, Cc are *perpendicular* to the sides of the triangle, we have

$$Ab = AB \cos A,\quad Bc = BC \cos B,\quad Ca = AC \cos C:$$

$$Ac = AC \cos A,\quad Ba = AB \cos B,\quad Cb = BC \cos C:$$

$$\therefore\ Ab.Bc.Ca = Ac.Ba.Cb:$$

and the perpendiculars have therefore a common point.

(3) If Aa, Bb, Cc, *bisect* the corresponding angles,

$$Ab = \frac{Bb \sin \frac{1}{2}B}{\sin A},\quad Bc = \frac{Cc \sin \frac{1}{2}C}{\sin B},\quad Ca = \frac{Aa \sin \frac{1}{2}A}{\sin C}:$$

$$Ac = \frac{Cc \sin \frac{1}{2}C}{\sin A},\quad Ba = \frac{Aa \sin \frac{1}{2}A}{\sin B},\quad Cb = \frac{Bb \sin \frac{1}{2}B}{\sin C}:$$

$\therefore\ Ab.Bc.Ca = Ac.Ba.Cb$, and the conclusion holds.

Similarly of other cases, and the second condition may be used, when convenient, to establish the same.

34. To find the greatest triangle described upon a given base, and having a given perimeter.

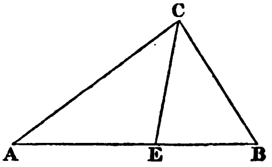
Let $2p$ be the given base, and $2q$ the sum of the remaining sides: then if $q+x$ be one of these sides, $q-x$ will be the other: whence we shall have

$$S=\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\sqrt{(q+p)(q-p)(p+x)(p-x)}=\sqrt{(q^2-p^2)(p^2-x^2)},$$

which will be the *greatest possible* when $x=0$, or when the two remaining sides are *equal:* that is, the *isosceles* triangle, whose sides are each $=q$, and whose area $=p\sqrt{q^2-p^2}$, is the triangle required.

35. If an angle of a triangle be bisected by a line cutting the opposite side, find the segments of the side and the bisecting line.



Here, $1=\dfrac{\sin ACE}{\sin BCE}=\dfrac{\sin ACE}{\sin AEC}\cdot\dfrac{\sin BEC}{\sin BCE}=\dfrac{BC}{AC}\cdot\dfrac{AE}{BE}$,

$\therefore \dfrac{AE}{BE}=\dfrac{AC}{BC}$: whence, $AE=\dfrac{bc}{a+b}$, and $BE=\dfrac{ac}{a+b}$.

Also, $CE^2=AC^2+AE^2-2AC\,.\,AE\cos A$

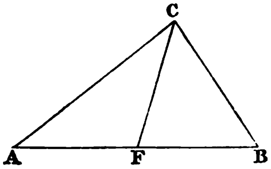
$$=b^2+\frac{b^2c^2}{(a+b)^2}-\frac{2b^2c}{a+b}\left(\frac{b^2+c^2-a^2}{2bc}\right)=ab-\frac{abc^2}{(a+b)^2}:$$

or, $$=\frac{ab(a+b+c)(a+b-c)}{(a+b)^2}=\frac{4a^2b^2}{(a+b)^2}\cos^2\tfrac{1}{2}C.$$

Whence, $ab=\dfrac{abc^2}{(a+b)^2}+CE^2=\left(\dfrac{ac}{a+b}\right)\left(\dfrac{bc}{a+b}\right)+CE^2$;

or, $AC\,.\,BC=AE\,.\,EB+CE^2$.

36. If a side of a triangle be bisected by a line drawn from the opposite angle, find the segments of the angle, and the bisecting line.



Here, $\frac{\sin ACF}{\sin CAF} = \frac{AF}{FC} = \frac{BF}{FC} = \frac{\sin BCF}{\sin CBF}$:

or, the sines of the segments of the vertical angle are proportional to the sines of the corresponding angles at the base: whence we have

$$\frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2}(ACF+BCF)}{\tan \frac{1}{2}(ACF-BCF)}:$$

from which $\frac{1}{2}(ACF-BCF)$ is found:

$$\therefore\ ACF = \tfrac{1}{2}(ACF+BCF) + \tfrac{1}{2}(ACF-BCF),$$

$$\text{and}\ \ BCF = \tfrac{1}{2}(ACF+BCF) - \tfrac{1}{2}(ACF-BCF).$$

Again, $CF^2 = CA^2 + AF^2 - 2AC\,.\,AF\cos A$

$$= b^2 + (\tfrac{1}{2}c)^2 - bc\cos A = \frac{2a^2+2b^2-c^2}{4}.$$

Since, $4CF^2 = 2a^2 + 2b^2 - c^2 = 2AC^2 + 2BC^2 - AB^2$,

we have $4CF^2 + 4AF^2 = 2AC^2 + 2BC^2$,

or, $AC^2 + BC^2 = 2AF^2 + 2CF^2$.

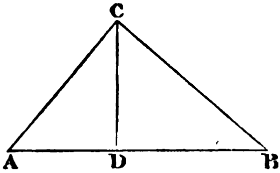
If the lengths of the three lines drawn from the angles to bisect the opposite sides, be denoted by h, k, l, we shall have

$$4(h^2+k^2+l^2) = 3(a^2+b^2+c^2),\ 16(h^2k^2+h^2l^2+k^2l^2)$$

$$= 9(a^2b^2+a^2c^2+b^2c^2),\ \text{and}\ 16(h^4+k^4+l^4) = 9(a^4+b^4+c^4):$$

from which it follows, by Article (76), that the area of the triangle whose *sides are* h, k, l is equal to $\frac{3}{4}$ of that of *the triangle, whose sides* are a, b, c.

37. To express a side of a triangle, in terms of the remaining sides and their opposite angles.



$$\text{Here, } AB = AD + DB = AC\cos A + BC\cos B,$$

$$\text{or, } c = b\cos A + a\cos B:$$

$$\text{similarly, } b = a\cos C + c\cos A:$$

$$\text{and } a = b\cos C + c\cos B.$$

38. Cor. From these results we may easily derive some of the fundamental Propositions of the Science.

$$\text{Thus, } \frac{c}{a} = \frac{b}{a}\cos A + \cos B: \text{ also, } \frac{b}{a} = \frac{\sin B}{\sin A},$$

$$\text{and } \frac{c}{a} = \frac{\sin C}{\sin A} = \frac{\sin\{180^\circ - (A+B)\}}{\sin A} = \frac{\sin(A+B)}{\sin A}:$$

$$\text{whence, } \frac{\sin(A+B)}{\sin A} = \frac{\sin B}{\sin A}\cos A + \cos B:$$

$$\text{or, } \sin(A+B) = \sin A\cos B + \cos A\sin B.$$

Again, by multiplying both members of the three equations above given by c, b, a respectively, we obtain

$$c^2 = bc\cos A + ac\cos B:$$

$$b^2 = ab\cos C + bc\cos A:$$

$$a^2 = ab\cos C + ac\cos B:$$

$$\therefore\ a^2 + b^2 + c^2 = 2ab\cos C + 2ac\cos B + 2bc\cos A:$$

$$\text{also, } 2a^2 = 2ab\cos C + 2ac\cos B:$$

$$\therefore\ b^2 + c^2 - a^2 = 2bc\cos A, \text{ and } \cos A = \frac{b^2 + c^2 - a^2}{2bc}:$$

$$\text{similarly, } \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \text{ and } \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

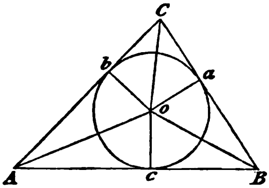
These conclusions being here arrived at without reference to *Geometry*, we shall be at liberty to use them n the establishment of any of its fundamental Propoitions: and it hence appears that *all* the formulæ of Trionometry may be derived from *one* relation.

Thus, let $C = 90^\circ$, and $\therefore \cos C = 0$: and we have $^2 = a^2 + b^2$, which is Prop. 47 of *Euclid* I.

Again, if $C = 60^\circ$, and $\therefore \cos C = \frac{1}{2}$, we find $c^2 = \dfrac{a^3 + b^3}{a + b}$:

if $C = 120^\circ$, and $\therefore \cos C = -\frac{1}{2}$, we get $c^2 = \dfrac{a^3 - b^3}{a - b}$.

39. To find the segments of the sides of a triangle, nade by the points of contact of the inscribed circle.



Here, it is evident from the construction that

$$Ac = Ab,\ Bc = Ba,\ Ca = Cb:$$

$$\therefore\ 2s = 2Ac + 2Bc + 2Ca = 2c + 2Ca:$$

whence, we have $Ca = s - c = Cb$:

similarly, $Ba = s - b = Bc$: and $Ac = s - a = Ab$.

Hence, $Ao^2 = Ac^2 + co^2 = (s-a)^2 + r^2 = bc\left(\dfrac{b+c-a}{b+c+a}\right)$.

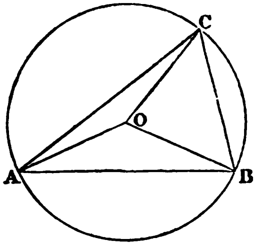
Also, $r = (s-a)\tan\frac{1}{2}A = (s-b)\tan\frac{1}{2}B =$ &c.:

and $S = \sqrt{s(s-a)(s-b)(s-c)}$

$$= (s-a)(s-b)\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = Ac\,.\,Bc\cot\tfrac{1}{2}C.$$

From Article (32. *Ap.*), it therefore follows that the hree straight lines joining the angular points with the oints of contact of the inscribed circle, pass through the ame point.

40. To find the segments of the angles of a triangle, made *by the radii of* the circumscribed circle.



$$\text{Here, } \angle\, OAB = \tfrac{1}{2}(180^\circ - AOB) = 90^\circ - C = \angle\, OBA:$$
$$\angle\, OAC = \tfrac{1}{2}(180^\circ - AOC) = 90^\circ - B = \angle\, OCA:$$
$$\angle\, OBC = \tfrac{1}{2}(180^\circ - BOC) = 90^\circ - A = \angle\, OCB.$$

41. Hence we may find the magnitude of the angle contained between the radii of the inscribed and circumscribed circles, of the same triangle.

By referring to the preceding diagrams, we have,

for the inscribed circle, $\angle\, oAB = \tfrac{1}{2}A$:

for the circumscribed circle, $\angle\, OAB = 90^\circ - C$:

and the difference of these, or OAo

$$= 90^\circ - C - \tfrac{1}{2}A = 90^\circ - C - \tfrac{1}{2}(180^\circ - B - C) = \tfrac{1}{2}(B - C).$$

42. To express the distance between the centres of the inscribed and circumscribed circles, in terms of their radii.

Let o and O be the centres of the inscribed and circumscribed circles: draw Ao, AO, and Oo:

$$\text{then, } Oo^2 = AO^2 + Ao^2 - 2AO \,.\, Ao \cos OAo$$
$$= R^2 + \frac{r^2}{\sin^2 \frac{1}{2}A} - \frac{2Rr \cos OAo}{\sin \frac{1}{2}A}:$$

but, $\cos OAo = \cos(\tfrac{1}{2}B - \tfrac{1}{2}C)$, by the last article,

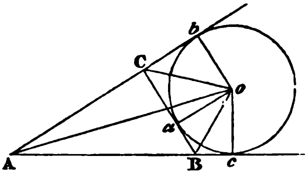
$$= \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C + \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C$$
$$= \frac{2s-a}{a}\sqrt{\frac{(s-b)(s-c)}{bc}} = \frac{2s-a}{a} \sin \tfrac{1}{2}A:$$
$$\therefore \frac{\cos OAo}{\sin \frac{1}{2}A} = \frac{2s-a}{a} = \frac{2(s-a)}{a} + 1:$$

also, $\frac{r^2}{\sin^2 \frac{1}{2}A} = \left(\frac{s-a}{s}\right) bc$, and $Rr = \frac{abc}{4s}$:

whence, $Oo^2 = R^2 + \frac{s-a}{s} bc - \frac{s-a}{s} bc - 2Rr$

$= R^2 - 2Rr$: and therefore $Oo = \sqrt{R^2 - 2Rr}$.

43. To find the radius of the circle touching one side of a triangle, and the two others produced.



Let $oa = ob = oc$ drawn from the centre o perpenlicular to the side BC, and to the sides AC, AB proluced, be denoted by r_1: join oA, oB and oC:

then, S = the area of the triangle ABC

= the trapezium $ABoC$ – the triangle BoC

$= \triangle AoC + \triangle AoB - \triangle BoC$

$= \frac{1}{2} r_1 (b + c - a)$:

from which we find $r_1 = \frac{2S}{b+c-a} = \frac{S}{s-a}$.

Similarly, if r_2, r_3, denote the radii of the circles ouching AC, AB, and the other sides produced respecively, we shall have $r_2 = \frac{S}{s-b}$, and $r_3 = \frac{S}{s-c}$.

Hence, $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s-a+s-b+s-c}{S} = \frac{s}{S} = \frac{1}{r}$:

nd $rr_1r_2r_3 = \frac{S^4}{s(s-a)(s-b)(s-c)} = S^2$, or $S = \sqrt{rr_1r_2r_3}$.

44. Cor. From $Ac = oc \cot \frac{1}{2}A$, we have $Ac = s = Ab$: nd from Article (43. *Ap.*), it appears that the distance etween the points of contact of the circles whose radii re r and r_1, with either produced side, is equal to the ide of the triangle, which lies between them.

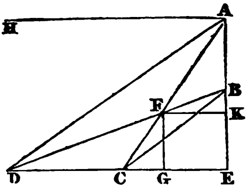
Also, it may be shewn, nearly as in Article (42. Ap.), *at the distances between* the centres of the circles whose radii are r_1, r_2, r_3, and the centre of the circumscribed circle whose radius is R, will be

$$\sqrt{R^2+2Rr_1},\ \sqrt{R^2+2Rr_2} \text{ and } \sqrt{R^2+2Rr_3}.$$

The circles forming the subject of these two articles, have been appropriately named the *Escribed* Circles of the triangle ABC: and the student is referred to the Appendix to the Ladies' Diary for 1835 and 1836, where he will find many curious and interesting results, in a series of papers entitled *Horæ Geometricæ*, by *T. S. Davies*, F.R.S. Professor of Mathematics at *Woolwich*.

V. *Mensuration of Heights and Distances.*

45. From a station D at the base of a mountain, a person observes its summit A at an elevation of 60°: after walking one mile towards the summit, along a plane making 30° with the horizon, to another station F, he finds the angle DFA to be 135°: determine the height of the mountain in yards.



Let ABE be perpendicular to the horizontal line DE at E: draw FG parallel and FK perpendicular to it: then $DF = 1$ mile:

$$\angle FDG = 30^\circ, \text{ and } \therefore FG = DF \sin 30^\circ = \tfrac{1}{2} \text{ mile}:$$

now in the triangle DFA, we have $DF = 1$ mile:

$$\angle ADF = \angle ADE - \angle FDG = 60^\circ - 30^\circ = 30^\circ:$$

$$\angle FAD = 180^\circ - (ADF + AFD) = 180^\circ - 165^\circ = 15^\circ:$$

$$\text{whence, } \frac{AF}{FD} = \frac{\sin 30^\circ}{\sin 15^\circ} = 2\cos 15^\circ, \text{ or } AF = 2\cos 15^\circ:$$

$$\text{also, } \angle FAK = \angle DAE - \angle DAF = 30^\circ - 15^\circ = 15^\circ:$$

$$\text{whence, } AK = AF \cos 15^\circ = 2\cos^2 15^\circ = \tfrac{1}{2}(2+\sqrt{3}) \text{ miles}:$$

$$\text{and } \therefore AE = AK + KE = AK + FG = \tfrac{1}{2}(3+\sqrt{3}) \text{ miles}$$

$$= 880\,(3+\sqrt{3}) \text{ yards} = 4164.6 \text{ yards, nearly.}$$

46. To find the height of an object standing upon a hill, contiguous to a horizontal plane.

Keeping in view the same diagram, let AB be the object; C, D two stations in the horizontal plane in a line with it: produce AB to meet the plane in E: at C observe the angles ACE, $BCE = C$ and C': suppose $CD = a$, and observe the angles ADE, $BDE = D$ and D':

$$\text{then, } \frac{AC}{CD} = \frac{\sin D}{\sin (C - D)}; \text{ or } AC = \frac{a \sin D}{\sin (C - D)}:$$

$$\text{and } \frac{BC}{CD} = \frac{\sin D'}{\sin (C' - D')}; \text{ or } BC = \frac{a \sin D'}{\sin (C' - D')}:$$

whence in the triangle ACB, we have found the two sides AC, BC; and the included angle $ACB = C - C'$ is given: and therefore the height AB may be determined by the ordinary process.

Precisely in the same manner, the distance between two objects at A and B, which are inaccessible to the observer at C and D, and to each other, may be ascertained.

47. From the top of an eminence of given height, the angles of depression of two objects on the horizon in the same vertical plane with it, are observed: to find the distance between them.

The same diagram being retained, let AE be the given eminence $= a$; C, D the objects in the horizon: draw AH parallel to the horizon, and let the angles of depression HAC, HAD of the objects be called α, β:

$$\text{then, } \frac{AC}{AE} = \frac{\sin AEC}{\sin ACE} = \frac{1}{\sin \alpha}; \text{ or } AC = \frac{a}{\sin \alpha}:$$

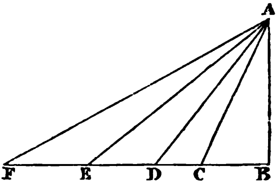
$$\text{and } \frac{CD}{CA} = \frac{\sin CAD}{\sin ADC} = \frac{\sin (\alpha - \beta)}{\sin \beta}:$$

$$\therefore\ CD = CA \frac{\sin (\alpha - \beta)}{\sin \beta} = \frac{a \sin (\alpha - \beta)}{\sin \alpha \sin \beta}, \text{ is found.}$$

Here also, the distances of the objects C, D from the point of observation A, are $a \operatorname{cosec} \alpha$, $a \operatorname{cosec} \beta$ respectively.

48. Given the distances between three stations in a straight line with an object, standing upon a horizontal

plane, and that the angles of its elevation at these stations are equal to θ, $90^\circ-\theta$ and 2θ in order, θ being unknown: to find its height.



Let AB be the object, E, D, C the stations in a straight line with it: $ED=a$, $DC=b$: then,

$$\angle AEB=\theta,\quad ADB=90^\circ-\theta,\ \text{ and } ACB=2\theta:$$

$$\therefore \angle EAC=\angle ACB-\angle AEB=\theta=\angle AEC,\ \text{and } EC=AC:$$

$$\text{also, } \angle EAD=\angle ADB-\angle AEB=90^\circ-2\theta:$$

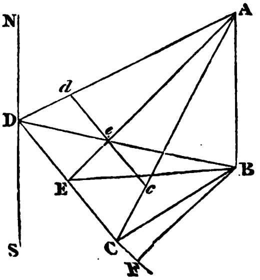
$$\text{whence, } \frac{AD}{DE}=\frac{\sin\theta}{\cos 2\theta},\ \text{and } \therefore AD=\frac{a\sin\theta}{\cos 2\theta}:$$

$$\text{again, } \frac{AC}{AD}=\frac{\cos\theta}{\sin 2\theta},\ \text{ or } AC=\frac{a\sin\theta}{\cos 2\theta}\,\frac{\cos\theta}{\sin 2\theta}=\frac{a}{2\cos 2\theta}:$$

$$\therefore \frac{a}{2\cos 2\theta}=EC=a+b,\ \text{which gives } \cos 2\theta=\frac{a}{2(a+b)}:$$

whence, $AB=AC\sin 2\theta=\frac{1}{2}\sqrt{4(a+b)^2-a^2}$, the required height.

49. To determine the height and distance of an object by means of its observed equal elevations, at two given points, and its observed elevation at the middle point between them.



Let AB be the object, $CD = a$, $CE = ED = \frac{1}{2}a$:

$$\alpha = \angle ACB = \angle ADB, \text{ and } \beta = \angle AEB:$$

$$\text{then, } \frac{\tan\alpha}{\tan\beta} = \frac{AB}{BC} \div \frac{AB}{BE} = \frac{BE}{BC} = \sin BCE:$$

$$\therefore BE = CE \tan BCE = \tfrac{1}{2}a \frac{\sin BCE}{\cos BCE} = \frac{a\tan\alpha}{2\sqrt{\tan^2\beta - \tan^2\alpha}}:$$

$$\text{whence, } AB = BE\tan\beta = \frac{a\tan\alpha\tan\beta}{2\sqrt{\tan^2\beta - \tan^2\alpha}}$$

$$= \frac{a\sin\alpha\sin\beta}{2\sqrt{\sin(\beta+\alpha)\sin(\beta-\alpha)}}, \text{ the height of the object;}$$

$$\text{also, } BC = AB\cot\alpha = \frac{a\cos\alpha\sin\beta}{2\sqrt{(\sin\beta+\alpha)\sin(\beta-\alpha)}} = BD:$$

$$\text{and } BE = AB\cot\beta = \frac{a\sin\alpha\cos\beta}{2\sqrt{\sin(\beta+\alpha)\sin(\beta-\alpha)}}.$$

50. Given the elevations of an object above a horizontal plane at three points, at given distances from one another, in the same straight line, to find its height.

In the preceding diagram, let AB be the object, C, E, D the three stations: $CE = a$, $ED = b$, $AB = h$: and let the observed elevations at C, E, D be α, β, γ: then,

$$BC = h\cot\alpha,\ BE = h\cot\beta,\ BD = h\cot\gamma:$$

draw BF perpendicular to DC, and we shall have

$$h^2\cot^2\alpha = a^2 + h^2\cot^2\beta + 2a\,.\,EF:$$

$$h^2\cot^2\gamma = b^2 + h^2\cot^2\beta - 2b\,.\,EF:$$

$$\therefore bh^2\cot^2\alpha = a^2b + bh^2\cot^2\beta + 2ab\,.\,EF:$$

$$ah^2\cot^2\gamma = ab^2 + ah^2\cot^2\beta - 2ab\,.\,EF:$$

$$\text{whence, } \{a\cot^2\gamma - (a+b)\cot^2\beta + b\cot^2\alpha\}\,h^2 = ab(a+b):$$

$$\therefore h = \sqrt{\frac{ab(a+b)}{a\cot^2\gamma - (a+b)\cot^2\beta + b\cot^2\alpha}}, \text{ the height:}$$

and thus, the values of BC, BE, BD are also determined.

The expression for h just found, though in a neat form as to algebraical symmetry, is not adapted to logarithms: but its value may be computed from the formula,

$$h^2 = \frac{a(a+b)\dfrac{\sin^2 \alpha \sin^2 \beta}{\sin(\beta+\alpha)\sin(\beta-\alpha)}}{1+\dfrac{a\sin^2\alpha \sin(\beta+\gamma)\sin(\beta-\gamma)}{b\sin^2\gamma \sin(\beta+\alpha)\sin(\beta-\alpha)}},$$

with the use of a subsidiary angle.

If we suppose $b=a$, and $\gamma=\alpha$, the problem is reduced to the next preceding.

51. Four objects situated at unequal but given distances, in the same straight line, appear to a spectator in the same plane with them, to be at equal distances from each other: it is required to determine his position.

In the diagram of Article (48. *Ap.*), let *C*, *D*, *E*, *F* be the objects, *A* the place of the eye: draw *AB* perpendicular to *FC*, and let $CD=a$, $DE=b$, $EF=c$:

also, let $\angle CAD = \angle DAE = \angle EAF = \phi$, and $\angle ACB = \theta$:

$$\text{then, } \frac{\sin 3\phi}{\sin ACF} = \frac{\sin CAF}{\sin ACF} = \frac{CF}{AF}:$$

$$\text{and } \frac{\sin ACD}{\sin \phi} = \frac{\sin ACD}{\sin CAD} = \frac{AD}{CD}:$$

$$\therefore \frac{\sin 3\phi}{\sin \phi} = 3 - 4\sin^2\phi = \frac{CF}{AF}\cdot\frac{AD}{CD} = \frac{CF}{CD}\cdot\frac{AL}{AF}$$

$$= \frac{CF}{CD}\cdot\frac{DE}{EF} = \frac{(a+b+c)b}{ac}:$$

$$\text{whence is found, } \sin\phi = \sqrt{\frac{3ac-(a+b+c)b}{4ac}}:$$

$$\text{again, } \sin\theta = \frac{AB}{AC}: \text{ and } \sin(\theta-2\phi) = \frac{AB}{AE}:$$

$$\therefore \frac{\sin\theta}{\sin(\theta-2\phi)} = \frac{AE}{AC} = \frac{ED}{DC} = \frac{b}{a}:$$

$$\text{and thence, } \tan\theta = \frac{b\sin 2\phi}{b\cos 2\phi - a} \text{ is determined:}$$

$$\text{also, } \frac{AC}{CD} = \frac{\sin(\theta-\phi)}{\sin\phi} \text{ gives } AC = \frac{a\sin(\theta-\phi)}{\sin\phi}:$$

$$\therefore BC = \frac{a\cos\theta\sin(\theta-\phi)}{\sin\phi}: \quad AB = \frac{a\sin\theta\sin(\theta-\phi)}{\sin\phi}:$$

so that the position of the point A, at which the spectator is situated, is completely determined.

If $(a+b+c)\,b$ be greater than $3ac$, the value of ϕ is *imaginary*, or the data are *incongruous*.

If $a=b=c$, we have $\sin\phi=0$, and BC, AB indefinitely great; which shews that *equidistant* objects in the same straight line, can never subtend *equal* angles at the eye of a spectator, at a *finite distance* from them.

52. Two ships are sailing in parallel directions, and a person in one of them observes the bearing of the other to be a° from the north: p hours afterwards its bearing was β°, and q hours after that, it was γ°: find the course of the vessels.

In the diagram of Article (49. *Ap.*), let SDN be the direction of the north: D, E, C the three positions of the observer; d, e, c the contemporaneous positions of the other vessel: then, since the motions are supposed to be uniform and in parallel directions, it is evident that Dd, Ee, Cc produced will all meet in some point A: let FDN the required angle be denoted by θ:

$$\therefore\ \angle FDA=\theta-a,\quad \angle FEA=\theta-\beta,\quad \angle FCA=\theta-\gamma:$$

$$\text{whence, } \angle DAE=a-\beta, \text{ and } \angle EAC=\beta-\gamma:$$

$$\text{and } \frac{DE}{EC}=\frac{DE}{EA}\cdot\frac{EA}{EC}=\frac{\sin DAE}{\sin ADE}\cdot\frac{\sin FCA}{\sin EAC}:$$

$$\text{or, } \frac{p}{q}=\frac{\sin(a-\beta)\sin(\theta-\gamma)}{\sin(\theta-a)\sin(\beta-\gamma)}:$$

$$\therefore\ \tan\theta=\frac{p\sin a\sin(\beta-\gamma)-q\sin\gamma\sin(a-\beta)}{p\cos a\sin(\beta-\gamma)-q\cos\gamma\sin(a-\beta)}.$$

If the observed vessel be *stationary*, the point A coincides with it, and the result will remain the same.

53. A flagstaff of given length, standing upon the top of a tower of given height, subtends at the eye of a spectator on the horizon an angle $=a^\circ$: find his distance from the tower.

In the diagram of Article (114. Ex. 3.), let $BF=a$, the height of the tower, and $FA=b$, the length of the staff: then, *if D be the position of* the spectator, $\angle ADF=a$:

$$\text{let } \angle BDF = \theta, \text{ and } \therefore \angle BDA = \theta + \alpha:$$

$$\therefore BD = BF \cot\theta = a \cot\theta:$$

$$\text{also, } BD = BA \cot(\theta+\alpha) = (a+b)\cot(\theta+\alpha):$$

$$\text{whence, we have } a\tan(\theta+\alpha) = (a+b)\tan\theta:$$

$$\text{which gives } \tan\theta = \frac{b\cot\alpha \pm \sqrt{b^2\cot^2\alpha - 4a(a+b)}}{2(a+b)}:$$

$$\text{and } \therefore BD = \frac{2a(a+b)}{b\cot\alpha \pm \sqrt{b^2\cot^2\alpha - 4a(a+b)}},$$

the required distance: and both the values will answer the conditions of the Problem.

In order that there may be no incongruity in the data, and to ensure a possible solution, it is evident that $b^2\cot^2\alpha$ must not be less than $4a(a+b)$: and the *extreme* case will therefore be, when $b^2\cot^2\alpha = 4a(a+b)$, and consequently α the *greatest* possible, so that

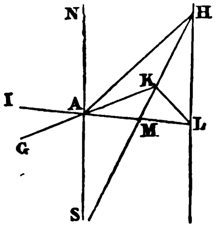
$$b\cot\alpha = 2\sqrt{a(a+b)}, \text{ and } \tan\theta = \sqrt{\frac{a}{a+b}}:$$

$$\text{and therefore } BD = \sqrt{a(a+b)} = \sqrt{BF.BA}.$$

This problem admits of a neat geometrical construction: for, if upon AF, a segment of a circle be described containing an angle $= \alpha^0$, it will cut the horizontal line in *two* points corresponding to the *two* values of BD above found: and when these two values become *equal* to each other, BD becomes a *tangent* to the circle at D, where ADF is the greatest possible, and $BD^2 = BF.BA$.

54. A person flying a kite at noon when the wind was blowing α^0 from the south, observed the angular distance of its shadow from the north to be β^0: but the wind having *suddenly* changed to α'^0 from the south, the shadow was found to be cast β'^0 from the north, and the elevation of the kite to be as much above 45°, as it was before below it: required the angular heights of the sun and kite.

Let A be the place of the spectator, K that of the kite, H that of its shadow on the horizon; draw SAN to represent the meridian, join AK, AH: draw KL perpendicular to the horizon, and join LH, AL and HK: then, since the shadow at noon is directly to the north of the object, LH is parallel to SAN: also, since AK



is the direction of the string, it is evident that the wind is blowing in the direction of the plane ALK, and

$$\therefore \angle LAN = \angle IAS = \alpha :$$

again, AH being the direction of the shadow from the spectator, we have $\angle NAH = \beta$, and $\therefore \angle HAL = \alpha - \beta$:

now, let the altitude of the sun, or $\angle LHK = \theta$:

and the altitude of the kite, or $\angle LAK = 45^\circ - \phi$:

$$\therefore \tan\theta = \frac{KL}{HL}, \text{ and } \tan(45^\circ - \phi) = \frac{KL}{AL} :$$

$$\text{whence, } \frac{\tan\theta}{\tan(45^\circ - \phi)} = \frac{AL}{HL} = \frac{\sin AHL}{\sin HAL} = \frac{\sin\beta}{\sin(\alpha - \beta)} :$$

$$\text{similarly, we have } \frac{\tan\theta}{\tan(45^\circ + \phi)} = \frac{\sin\beta'}{\sin(\alpha' - \beta')},$$

since the altitude of the sun remains the same, and $45^\circ - \phi$ is merely changed into $45^\circ + \phi$:

$$\text{whence, } \tan^2\theta = \frac{\sin\beta \sin\beta'}{\sin(\alpha - \beta)\sin(\alpha' - \beta')},$$

which gives the angular altitude of the sun :

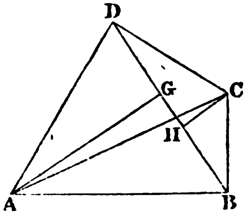
$$\text{and } \tan^2(45^\circ - \phi) = \frac{\sin(\alpha - \beta)\sin\beta'}{\sin(\alpha' - \beta')\sin\beta},$$

which determines the angular altitude of the kite to the spectator, when in its first position.

VI. *Properties of Quadrilaterals.*

55. To express the angles, diagonals, area, &c. of a quadrilateral, in terms of its sides, the opposite angles being *supplements to each* other.

Let $AB = a$, $BC = b$, $CD = c$, $DA = d$, $BD = h$, $AC = k$:



then, $2ad \cos A = a^2 + d^2 - h^2$, $2bc \cos C = b^2 + c^2 - h^2$:

but $\cos C = \cos (180^\circ - A) = - \cos A$:

$$\left.\begin{aligned} \therefore\ \cos A &= \frac{a^2 + d^2 - b^2 - c^2}{2(ad + bc)} = - \cos C: \\ \text{similarly, } \cos B &= \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)} = - \cos D. \end{aligned}\right\} \quad (1)$$

Hence, $\sin A = \dfrac{2}{ad + bc}\sqrt{(s-a)(s-b)(s-c)(s-d)}$,

where $2s = a + b + c + d$. And similarly of the rest. (2)

$$\left.\begin{aligned} \text{Also, } \sin \tfrac{1}{2}A &= \sqrt{\frac{(s-a)(s-d)}{ad+bc}} = \cos \tfrac{1}{2}C: \\ \cos \tfrac{1}{2}A &= \sqrt{\frac{(s-b)(s-c)}{ad+bc}} = \sin \tfrac{1}{2}C: \\ \tan \tfrac{1}{2}A &= \sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}} = \cot \tfrac{1}{2}C. \end{aligned}\right\} \quad (3)$$

Again, $\dfrac{a^2 + d^2 - h^2}{2ad} = - \dfrac{b^2 + c^2 - h^2}{2bc}$, gives

$$\left.\begin{aligned} h^2 &= \frac{(a^2 + d^2)bc + (b^2 + c^2)ad}{ad + bc} = \frac{(ac + bd)(ab + cd)}{ad + bc}: \\ \text{similarly, } k^2 &= \frac{(ac + bd)(ad + bc)}{ab + cd}. \end{aligned}\right\} \quad (4)$$

Hence, $hk = ac + bd$: or, in geometrical symbols, $AC \,.\, BD = AB \,.\, CD + BC \,.\, AD$: *Euclid* VI. Prop. D.

Also, $\dfrac{h}{k} = \dfrac{ab + cd}{ad + bc}$: or, $\dfrac{BD}{AC} = \dfrac{AB \,.\, BC + CD \,.\, DA}{AB \,.\, AD + BC \,.\, CD}$:

that is, the diagonals are to each other as the sums of the rectangles of the *conterminous* sides, respectively meeting their extremities.

The area of $ABCD = \triangle ABD + \triangle BCD$

$$= \tfrac{1}{2} ad \sin A + \tfrac{1}{2} bc \sin C = \tfrac{1}{2}(ad + bc) \sin A$$

$$= \sqrt{(s-a)(s-b)(s-c)(s-d)}. \qquad (5)$$

If θ be the angle contained between the diagonals,

$$ABCD = \triangle ABD + \triangle CBD = \tfrac{1}{2} BD(AG + CH)$$

$$= \tfrac{1}{2} BD \,.\, AC \sin \theta = \tfrac{1}{2} hk \sin \theta :$$

$$\text{and } \therefore \sin \theta = \frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ac + bd}. \qquad (6)$$

If R be the radius of the circle circumscribed about the triangle ABD, we have

$$R = \frac{had}{4 \triangle ABD} = \frac{had}{2\, ad \sin A} = \frac{h}{2 \sin A} = \frac{h}{2 \sin C} :$$

$$\text{also, } = \frac{k}{2 \sin B} = \frac{k}{2 \sin D} :$$

from which it appears that the circle, whose radius is R, will be circumscribed about the quadrilateral: and

$$R = \frac{h}{2 \sin A} = \frac{h(ad + bc)}{4\sqrt{(s-a)(s-b)(s-c)(s-d)}}$$

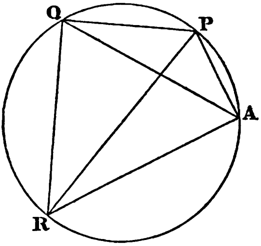
$$= \frac{1}{4} \sqrt{\frac{(ab + cd)(ac + bd)(ad + bc)}{(s-a)(s-b)(s-c)(s-d)}}. \qquad (7)$$

Hence, with any four lines, each of which is less than the sum of the others, it is possible to construct a trapezium which may be inscribed in a circle: and though the *area* will always be the same in whatever order the lines are taken, the *angles*, &c. will generally be different.

If the side $d = 0$, all these formulæ become the same as those already established for a triangle.

56. To find the relations between the sides of a quadrilateral, which is capable of being inscribed in one circle, and circumscribed about another.

Let $AP = a$, $PQ = b$, $QR = c$, $RA = d$, $AQ = h$, $PR = k$: then, *because* the quadrilateral is inscribed in a circle,



$S = \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where $2s = a+b+c+d$: but since it may be circumscribed about a circle, we have likewise $a + c = b + d$, as in Article (39. *Ap.*): whence,

$$s-a=c, \text{ and } \therefore s-c=a: \; s-b=d, \text{ and } \therefore s-d=b:$$

$$\therefore S = \sqrt{abcd} = \sqrt{ab} \times \sqrt{cd}, \text{ or } = \sqrt{ac} \times \sqrt{bd}:$$

that is, the area of the quadrilateral is equal to the rectangle contained by the two mean proportionals between the pairs of *adjacent*, or *opposite* sides:

$$\text{now, } S = \sqrt{ab} \times \sqrt{cd} = \sqrt{ac} \times \sqrt{bd},$$

comprises both the prescribed conditions, and it only remains to determine in what ways the equation $a + c = b + d$, can be satisfied: and this may evidently be done as follows.

(1) When $b = a$, and $d = c$.

Here, we have $S = \sqrt{abcd} = ac$:

$$\sin A = \frac{2S}{ad+bc} = \frac{2ac}{ac+ac} = 1: \; \therefore \angle A = 90^\circ = \angle Q:$$

$$\text{also, } \sin P = \frac{2S}{ab+cd} = \frac{2ac}{a^2+c^2} = \sin R.$$

(2) When $d = a$, and $b = c$.

Here, we have $S = \sqrt{abcd} = ac$:

$$\text{and } \sin A = \frac{2S}{ad+bc} = \frac{2ac}{a^2+c^2} = \sin Q:$$

$$\textit{also}, \; \sin P = \frac{2S}{ab+cd} = \frac{2ac}{ac+ac} = 1: \; \therefore \angle P = 90^\circ = \angle R.$$

And in both cases, $\sin\theta = \dfrac{2S}{ac+bd} = 1$, or $\theta = 90°$, and the diagonals intersect each other at *right* angles.

We conclude then that the quadrilateral must consist of two equal right-angled triangles having a common hypothenuse, which is therefore the diameter of the circumscribed circle: and that it may also be divided into two isosceles triangles with a common base, which cuts the diameter at right angles.

The diameter of the circumscribed circle will evidently be $= \sqrt{a^2+d^2}$, or $\sqrt{b^2+c^2}$;

and that of the inscribed circle will be $= \dfrac{ac}{a+c}$, or $= \dfrac{bd}{b+d}$: also, $hk = 2ad$, or $= 2bc$, which is *twice* the area.

It need scarcely be mentioned that the same holds, when $a = b = c = d$, and the quadrilateral becomes a square.

57. To express the area of any quadrilateral in terms of its sides, and two of its opposite angles.

Using the diagram and notation of Article (55. *Ap.*) and denoting the area by S, we have

$$S = \tfrac{1}{2}(ad\sin A + bc\sin C):$$

$$\text{but, } \cos A = \frac{a^2+d^2-h^2}{2ad}, \text{ and } \cos C = \frac{b^2+c^2-h^2}{2bc}:$$

$$\therefore\ 2ad\cos A - 2bc\cos C = a^2+d^2-b^2-c^2:$$

whence, by the ordinary substitutions, we get

$$(a+d)^2-(b-c)^2 = 4ad\cos^2\tfrac{1}{2}A + 4bc\sin^2\tfrac{1}{2}C:$$

$$(b+c)^2-(a-d)^2 = 4ad\sin^2\tfrac{1}{2}A + 4bc\cos^2\tfrac{1}{2}C:$$

and by multiplying together the corresponding members of these equalities, and dividing by 16, we obtain

$$\begin{aligned}
&(s-a)(s-b)(s-c)(s-d)\\
&= a^2d^2\sin^2\tfrac{1}{2}A\cos^2\tfrac{1}{2}A + abcd\cos^2\tfrac{1}{2}A\cos^2\tfrac{1}{2}C\\
&\quad + b^2c^2\sin^2\tfrac{1}{2}C\cos^2\tfrac{1}{2}C + abcd\sin^2\tfrac{1}{2}A\sin^2\tfrac{1}{2}C\\
&= (ad\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A + bc\sin\tfrac{1}{2}C\cos\tfrac{1}{2}C)^2\\
&\quad + abcd(\cos\tfrac{1}{2}A\cos\tfrac{1}{2}C - \sin\tfrac{1}{2}A\sin\tfrac{1}{2}C)^2\\
&= \tfrac{1}{4}(ad\sin A + bc\sin C)^2 + abcd\cos^2\tfrac{1}{2}(A+C)\\
&= S^2 + abcd\cos^2\tfrac{1}{2}(A+C):
\end{aligned}$$

$$\therefore\ S^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\tfrac{1}{2}(A+C).$$

Since $A+B+C+D=360^\circ$, we have $\frac{1}{2}(A+C)$ $=180^\circ - \frac{1}{2}(B+D)$, and $\cos^2\frac{1}{2}(A+C) = \cos^2\frac{1}{2}(B+D)$: and therefore the area is expressed in terms of the sides, and *either* pair of opposite angles, by the same formula.

The diagonals and the remaining angles, may easily be expressed by means of the same data:

also, if $A+C=B+D=180^\circ$,

we have $\cos^2\frac{1}{2}(A+C) = \cos^2\frac{1}{2}(B+D) = 0$:

and the result (5) of Article (55. *Ap.*) is obtained.

Hence, it appears that four given lines taken in any order will comprise the *greatest* area, when they form a trapezium capable of being inscribed in a circle.

VII. *General Properties of Polygons.*

58. Any side of a polygon, is equal to the sum of the products of each of the other sides and the cosine of its interior angle of inclination to that side.

Draw any straight line AX, and let the sides of the polygon $ABCDE$ &c. taken in order from the point A, and lying *above* AX be called a, b, c, &c. and make the angles α, β, γ, &c. with it: then, if x be the base of the polygon in the line AX, we shall evidently have

$$x = a\cos\alpha + b\cos\beta + c\cos\gamma + \&c.$$

exactly as for the triangle in Article (37. *Ap.*).

Hence, it will also appear that the perpendicular let fall from any angle of a polygon upon the base, is equal to the sum of the products of each side, comprised between that angle and the base, and the sine of its inclination to the base.

59. The square of a side of any polygon, is equal to the sum of the squares of all the other sides, diminished by twice the sum of the products of all these sides taken two and two together, and the cosines of their included angles.

For, let the sides be a, b, c, d, &c. taken in order: *and let a* make with the sides that follow it, the angles α_1, α_2, α_3, &c.: b with those that follow it, the angles β_1,

β_3, &c.: and so on for the rest: then by the last cle, we have

$$a = b\cos\alpha_1 + c\cos\alpha_2 + d\cos\alpha_3 + \&c.:$$
$$b = a\cos\alpha_1 + c\cos\beta_1 + d\cos\beta_2 + \&c.:$$
$$c = a\cos\alpha_2 + b\cos\beta_1 + d\cos\gamma_1 + \&c.:$$
$$d = a\cos\alpha_3 + b\cos\beta_2 + c\cos\gamma_1 + \&c.: \&c.$$
$$\therefore\ a^2 = ab\cos\alpha_1 + ac\cos\alpha_2 + ad\cos\alpha_3 + \&c.:$$
$$b^2 = ab\cos\alpha_1 + bc\cos\beta_1 + bd\cos\beta_2 + \&c.:$$
$$c^2 = ac\cos\alpha_2 + bc\cos\beta_1 + cd\cos\gamma_1 + \&c.:$$
$$d^2 = ad\cos\alpha_3 + bd\cos\beta_2 + cd\cos\gamma_1 + \&c.: \&c.$$

nce from the first, subtracting all the rest and transing, we obtain

$$a^2 = b^2 + c^2 + d^2 + \&c.$$
$$-2\,(bc\cos\beta_1 + bd\cos\beta_2 + \&c. + cd\cos\gamma_1 + \&c.).$$

he principle of this process has already been applied triangle in Article (37. *Ap.*).

For other propositions connected with this subject, reader is referred to a treatise entitled *Polygonome-, ou de la mesure des Figures rectilignes, par Simon ilier :* or, to an extract from it, contained in the third me of *Dr Hutton's* Course of Mathematics.

VIII. *Problems on Regular Polygons.*

60. To find the perimeter and area of a polygon ^{m} sides, inscribed in a circle of given radius.

From Article (82), we find the perimeter of the poly- to be $= 2^{m+1} r \sin\dfrac{180^\circ}{2^m}$:

$$t,\ \sin\frac{1}{2}A = \frac{1}{2}\sqrt{2 - 2\cos A} = \frac{1}{2}\sqrt{2 - 2\sqrt{1 - \sin^2 A}}:$$
$$\sin\frac{1}{2^2}A = \frac{1}{2}\sqrt{2 - 2\sqrt{1 - \sin^2 \tfrac{1}{2}A}}:$$
$$\sin\frac{1}{2^3}A = \frac{1}{2}\sqrt{2 - 2\sqrt{1 - \sin^2 \frac{1}{2^2}A}}:\ \&c:$$
$$\sin\frac{1}{2^m}A = \frac{1}{2}\sqrt{2 - 2\sqrt{1 - \sin^2 \frac{1}{2^{m-1}}A}}:$$

whence, by continued substitution, we obtain $\sin\frac{180^\circ}{2^m}$

$$=\tfrac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\&c.\mp\sqrt{2}}}}},$$

wherein the radical sign occurs $m-1$ times, because

$$\sqrt{1-\sin^2\tfrac{1}{2}(180^\circ)}=\sqrt{1-1}=0:$$

∴ the perimeter of the polygon will be expressed by

$$2^m r\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\&c.\mp\sqrt{2}}}}}.$$

Also, the area of the polygon $= 2^{m-1}r^2\sin\frac{360^\circ}{2^m}$

$= 2^m r^2 \sin\frac{180^\circ}{2^m}\cos\frac{180^\circ}{2^m}$, is readily found.

In the same manner may be computed the perimet and area of the circumscribed polygon of 2^m sides, l means of the formulæ,

$$2^{m+1}r\tan\frac{180^\circ}{2^m}, \text{ and } 2^m r^2\tan\frac{180^\circ}{2^m}.$$

61. If the radius of the circle be 1, we should ha the following results from these formulæ.

Number of Sides.	Area of Inscribed Polygon.	Area of Circumscribed Polygon.
4	2.	4.
8	2.8284271	3.3137085
16	3.0614674	3.1825979
32	3.1214451	3.1517249
64	3.1365485	3.1441184
128	3.1403311	3.1422236
256	3.1412772	3.1417504
512	3.1415138	3.1416321
1024	3.1415729	3.1416025
2048	3.1415877	3.1415951
4096	3.1415914	3.1415933
8192	3.1415923	3.1415928
16384	3.1415925	3.1415927
32768	3.1415926	3.1415926

In this way it was that the ancient Mathematicians approximated to the value of the area of a circle, by continually *doubling* the number of sides of the inscribed and circumscribed regular polygons: and we perceive that the areas of the inscribed and circumscribed polygons of 32768 sides differ by no significant figure in the first *seven* places of decimals, and consequently that the area of the circle, which is always intermediate in magnitude to these two quantities, is correctly expressed by 3.1415926, to the *same extent*.

Also, since the area of a circle $= \frac{1}{2}$ circumference × radius, we shall have the circumference of a circle whose radius is 1, expressed by 6.2831852, nearly, and therefore $\pi = 3.1415926$, nearly.

62. To find the numbers of regular figures of the same kinds which, by their summits meeting at a point, can fill up the angular space about it.

Let n be the number of sides of each: then, at the centre of the figure, each side subtends an angle $= \frac{2\pi}{n}$: and therefore the angle at each summit

$$= \pi - \frac{2\pi}{n} = \frac{n-2}{n}\pi:$$

whence, to answer the proposed condition, if m be the number of figures required, we must have

$$m\left(\frac{n-2}{n}\right)\pi = 2\pi, \quad \text{or} \quad m = \frac{2n}{n-2};$$

$$\text{and } \therefore m - 2 = \frac{4}{n-2}, \text{ must be a whole number:}$$

that is, the only values which n admits of, are 3, 4 and 6: and therefore the only figures possessing the prescribed property, are *Equilateral Triangles, Squares* and *Hexagons,* whose numbers are 6, 4 and 3.

63. Cor. If three equal plane spaces be divided into the same number of equal regular figures as above described, it is evident that the area of one figure of each kind, will be of the same magnitude: and that the sum of the dividing lines will, in each case, be proportional to the length of a side of one of the regular figures, multiplied by the number of its sides.

Let t, s, h represent respectively a side of an equilateral triangle, square and hexagon, with the condition above given: T, S, H the sum of the dividing lines: then, since, if a be the length of a side, the area A of a regular polygon of n sides $= \frac{1}{4} na^2 \cot \frac{180^\circ}{n}$, we have

$$\text{if } n = 3,\ t = \frac{2\sqrt{A}}{\sqrt[4]{3}},\ \text{and } T = 2\sqrt[4]{27}\sqrt{A}:$$

$$\text{if } n = 4,\ s = \sqrt{A},\ \text{and } S = 4\sqrt{A}:$$

$$\text{if } n = 6,\ h = \frac{\sqrt{2A}}{\sqrt[4]{27}},\ \text{and } H = 2\sqrt[4]{12}\sqrt{A}:$$

$$\therefore\ T : S : H = \sqrt[4]{27} : \sqrt[4]{16} : \sqrt[4]{12}.$$

Hence it appears that H is the least of the three quantities T, S, H: and consequently that the sum of the dividing lines is the *least possible*, when the proposed area is divided into *hexagons*.

Thus, for a piece of *Net-work* of given extent, *the least quantity* of material will be requisite, when the meshes are arranged in the forms of *regular hexagons*. The same will evidently hold with respect to the division of an indefinite *solid space* into *right prisms*, of a given height, the dividing surfaces being the least possible, when their bases are regular hexagons; and this is precisely the case in the structure of *Honeycomb*, in which the cells are regular *six-sided* prisms, and are therefore so arranged that a given quantity of honey requires the least possible quantity of wax to contain it.

64. Given the areas a, A, of the regular polygons of n sides, inscribed in, and circumscribed about, a circle; to find those of the regular polygons of $2n$ sides, relative to the same circle.

Let a', A' be the required areas: then, we have

$$a = \tfrac{1}{2} nr^2 \sin \frac{360^\circ}{n},\quad A = nr^2 \tan \frac{180^\circ}{n}:$$

$$a' = nr^2 \sin \frac{180^\circ}{n},\quad A' = 2nr^2 \tan \frac{90^\circ}{n}:$$

to eliminate the quantities r and n:

$$\text{now, } \frac{a}{A} = \cos^2 \frac{180^\circ}{n},\ \text{and } \frac{a}{a'} = \cos \frac{180^\circ}{n}:$$

$$\therefore \left(\frac{a'}{a}\right)^2 = \frac{A}{a}, \text{ which gives } a' = \sqrt{Aa}:$$

$$\text{and } \frac{A'}{A} = \frac{\cos\frac{180^\circ}{n}}{\cos^2\frac{90^\circ}{n}} = \frac{2\cos\frac{180^\circ}{n}}{1+\cos\frac{180^\circ}{n}} = \frac{2a}{a'+a}$$

$$= \frac{2a}{\sqrt{Aa}+a}, \text{ which gives } A' = \frac{2A\sqrt{a}}{\sqrt{A}+\sqrt{a}}.$$

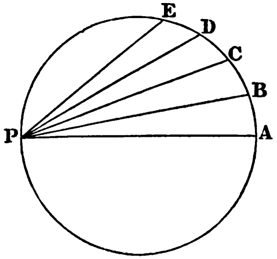
From the equality, $a' = \sqrt{Aa}$, we conclude that the area of a regular polygon of an even number of sides, inscribed in a circle, is a *mean proportional* between the areas of an inscribed and of a circumscribed regular polygon of half the number of sides:

and from the equality, $A' = \frac{2A\sqrt{a}}{\sqrt{A}+\sqrt{a}} = \frac{2A\sqrt{Aa}}{A+\sqrt{Aa}} = \frac{2Aa'}{A+a'}$,

it appears that the area of a regular polygon of an even number of sides, circumscribed about a circle, is a *harmonic mean* between the areas of an inscribed regular polygon of the same number of sides, and of a circumscribed regular polygon of half that number.

IX. *Properties of the Circle.*

65. The following property of the chords of a circle is said to have been discovered by *Vieta*, a very celebrated *French* Mathematician, born in the year 1540: but it has sometimes been attributed to *Dr. Waring,* who was elected *Lucasian Professor* of Mathematics at *Cambridge*, in the year 1760.



Let PA be the diameter of the circle, and let

$$AB = BC = CD = DE = \&c.:$$

then, if the chords PB, PC, PD, PE, &c. be drawn, and PB be assumed $= x + \frac{1}{x}$: it is required to prove that

$$PC = x^2 + \frac{1}{x^2},\quad PD = x^3 + \frac{1}{x^3},\quad PE = x^4 + \frac{1}{x^4},\ \text{\&c.}$$

Here,

$$PB = \text{chd}\,(\pi - AB) = 2\sin(\tfrac{1}{2}\pi - \tfrac{1}{2}AB) = 2\cos\tfrac{1}{2}AB = x + \frac{1}{x}:$$

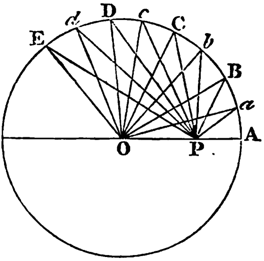
$$PC = \text{chd}\,(\pi - AC) = 2\sin(\tfrac{1}{2}\pi - \tfrac{1}{2}AC) = 2\cos\tfrac{1}{2}AC$$

$$= 2\cos 2(\tfrac{1}{2}AB) = x^2 + \frac{1}{x^2},\ \text{by Article (122):}$$

$$PD = \text{chd}\,(\pi - AD) = 2\sin(\tfrac{1}{2}\pi - \tfrac{1}{2}AD) = 2\cos\tfrac{1}{2}AD$$

$$= 2\cos 3(\tfrac{1}{2}AB) = x^3 + \frac{1}{x^3}:\ \text{and so on.}$$

66. The following is termed *Demoivre's* property of the circle.



Let P be any point within or without the circle, whose centre is O, and radius $OA = 1$; and let $BC = CD = DE = \text{\&c.} = \frac{2\pi}{n}$: and complete the construction:

then, if $OP = x$, and $\angle POB = \frac{a}{n}$, we have

$$x^{2n} - 2\cos a x^n + 1 = OP^{2n} - 2OP^n . OB^n \cos n\,POB + OB^{2n}:$$

$$x^2 - 2\cos\frac{a}{n}x + 1 = OP^2 - 2OP . OB\cos POB + OB^2 = PB^2:$$

$$x^2 - 2\cos\frac{2\pi + a}{n}x + 1 = OP^2 - 2OP . OC\cos POC + OC^2 = PC^2:$$

$$x^2 - 2\cos\frac{4\pi + a}{n}x + 1 = OP^2 - 2OP . OD\cos POD + OD^2 = PD^2:$$

$$\text{\&c.} \qquad = \text{\&c.} \qquad\qquad = \text{\&c.}:$$

whence, by Article (159), we shall obtain

$$OP^{2n} - 2OP^n . OB^n \cos n\, POB + OB^{2n} = PB^2 . PC^2 . PD^2 . \&c.$$

to n factors.

67. Cor. If P be in the circumference of the circle whose radius is r, we have $OP = r$: and the equality just investigated becomes

$$2r^n \sin \tfrac{1}{2} n\, POB = PB\,.\,PC\,.\,PD\,.\,\&c. \text{ to } n \text{ factors,}$$

which is the product of all the lines drawn to the angular points of a regular polygon, from any point in the circumference of its circumscribed circle.

68. If OP meet the circle in A, and $AB = BC = CD = DE = \&c. = \frac{2\pi}{n}$: we have

$$x^{2n} - 2x^n + 1 = OP^{2n} - 2OP^n . OA^n + OA^{2n} = (OP^n \sim OA^n)^2:$$

$$x^2 - 2x + 1 = OP^2 - 2OP\,.\,OA + OA^2 \quad = PA^2:$$

$$x^2 - 2\cos\frac{2\pi}{n}\, x + 1 = OP^2 - 2OP\,.\,OB \cos POB + OB^2 = PB^2:$$

$$x^2 - 2\cos\frac{4\pi}{n}\, x + 1 = OP^2 - 2OP\,.\,OC \cos POC + OC^2 = PC^2:$$

$$x^2 - 2\cos\frac{6\pi}{n}\, x + 1 = OP^2 - 2OP\,.\,OD \cos POD + OD^2 = PD^2:$$

$$\&c. \qquad = \&c. \qquad\qquad = \&c.:$$

whence, by the theorem above mentioned, we obtain

$$OP^n \sim OA^n = PA\,.\,PB\,.\,PC\,.\,PD\,.\,\&c. \text{ to } n \text{ factors.}$$

Again, if the arcs AB, BC, CD, &c. be bisected in a, b, c, &c., and Pa, Pb, Pc, &c., Oa, Ob, Oc, &c., be drawn, we shall have, as before,

$$\begin{aligned} OP^{2n} \sim OA^{2n} &= PA\,.\,Pa\,.\,PB\,.\,Pb\,.\,PC\,.\,Pc\,.\,\&c. \text{ to } 2n \text{ factors,} \\ &= (PA\,.\,PB\,.\,PC\,.\,\&c.)\,(Pa\,.\,Pb\,.\,Pc\,.\,\&c.) \\ &= (OP^n \sim OA^n)\,(Pa\,.\,Pb\,.\,Pc\,.\,\&c.): \end{aligned}$$

whence, $OP^n + OA^n = Pa\,.\,Pb\,.\,Pc\,.\,\&c.$ to n factors.

The discovery of these theorems is due to *Roger Cotes*, who was elected *Plumian Professor* of Astronomy and Experimental Philosophy at *Cambridge*, in the year 1707: and they are generally known by the name of *Cotes' Theorems* and *Properties of the Circle.*

69. Cor. 1. Hence, if a circle be described with the centre *O and radius* OP, and straight lines be drawn

from the point P, where it intersects any one of the radii from the centre of the circle $ABCD$ &c., to the angular points of a regular polygon of n sides inscribed in it, the continued product of these lines is equal to the invariable magnitude $OA^n - OP^n$.

Also, since $PA = OA - OP$, we shall have

$$PB \,.\, PC \,.\, PD \,.\, \text{\&c. to } n-1 \text{ factors} = \frac{OA^n - OP^n}{OA - OP}$$

$$= OA^{n-1} + OP \,.\, OA^{n-2} + OP^2 \,.\, OA^{n-3} + \text{\&c. to } n \text{ terms}:$$

and therefore when $OP = OA$, or P coincides with A, we get $AB \,.\, AC \,.\, AD \,.\,$ &c. to $n-1$ factors $= nOA^{n-1} = nr^{n-1}$: that is, the continued product of the lines drawn from any one angle of a regular inscribed polygon of n sides, to each of the other angles, is equal to nr^{n-1}.

Again, the equality, $Aa \,.\, Ab \,.\, Ac \,.\,$ &c. to n factors $= 2r^n$, shews that, if straight lines be drawn from an angular point of any regular inscribed polygon of an even number $2n$ of sides, to every alternate angle, beginning with the nearest, their continued product is equal to $2r^n$.

70. Cor. 2. Since $AB = \text{chd}\,\frac{2\pi}{n} = 2 \sin \frac{\pi}{n}$,

$AC = 2 \sin \frac{2\pi}{n}$, $AD = 2 \sin \frac{3\pi}{n}$, &c., we have

$$2^{n-1} \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \sin \frac{3\pi}{n} \text{ \&c. to } n-1 \text{ factors} = nr^{n-1}:$$

$$\text{and } \therefore \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \sin \frac{3\pi}{n} \text{ \&c. to } n-1 \text{ factors} = n\left(\frac{r}{2}\right)^{n-1}.$$

Also, since $Aa = \text{chd}\,\frac{\pi}{n} = 2 \sin \frac{\pi}{2n}$,

$Ab = 2 \sin \frac{3\pi}{2n}$, $Ac = 2 \sin \frac{5\pi}{2n}$, &c., we find

$$\sin \frac{\pi}{2n} \sin \frac{3\pi}{2n} \sin \frac{5\pi}{2n} \text{ \&c. to } n \text{ factors} = 2\left(\frac{r}{2}\right)^n.$$

Ex. If $n = 18$, we shall immediately obtain

$$\sin 5^\circ \sin 15^\circ \text{ \&c. } \sin 95^\circ \sin 105^\circ \text{ \&c. } \sin 175^\circ$$

$$= (\sin 5^\circ \sin 15^\circ \sin 25^\circ \text{ \&c. } \sin 85^\circ)^2 = \frac{1}{2^{17}}:$$

$$\text{and } \therefore \sin 5^\circ \sin 15^\circ \sin 25^\circ \text{ \&c. } \sin 85^\circ = \frac{1}{256\sqrt{2}}.$$

X. *Summation of Trigonometrical Series.*

71. To find the sum of the sines of a series of angles in arithmetical progression,

$$\sin\alpha + \sin(\alpha+\delta) + \sin(\alpha+2\delta) + \&\text{c. to } n \text{ terms.}$$

Since, $2\sin\frac{1}{2}\delta\sin\alpha = \cos(\alpha - \frac{1}{2}\delta) - \cos(\alpha + \frac{1}{2}\delta)$:

$$2\sin\tfrac{1}{2}\delta\sin(\alpha+\delta) = \cos(\alpha+\tfrac{1}{2}\delta) - \cos(\alpha+\tfrac{3}{2}\delta):$$

$$2\sin\tfrac{1}{2}\delta\sin(\alpha+2\delta) = \cos(\alpha+\tfrac{3}{2}\delta) - \cos(\alpha+\tfrac{5}{2}\delta): \&\text{c.}:$$

$$2\sin\tfrac{1}{2}\delta\sin\{\alpha+(n-1)\delta\}$$

$$= \cos\{\alpha+\tfrac{1}{2}(2n-3)\delta\} - \cos\{\alpha+\tfrac{1}{2}(2n-1)\delta\}:$$

if we denote the sum by s, we have, by addition,

$$2\sin\tfrac{1}{2}\delta s = \cos(\alpha-\tfrac{1}{2}\delta) - \cos\{\alpha+\tfrac{1}{2}(2n-1)\delta\}$$

$$= 2\sin\{\alpha+\tfrac{1}{2}(n-1)\delta\}\sin\tfrac{1}{2}n\delta:$$

$$\text{and } \therefore s = \frac{\sin\{\alpha+\frac{1}{2}(n-1)\delta\}\sin\frac{1}{2}n\delta}{\sin\frac{1}{2}\delta}.$$

If δ be made equal to α, 2α, &c. in succession, we find

$$\sin\alpha + \sin 2\alpha + \&\text{c.} + \sin n\alpha = \frac{\sin\frac{1}{2}(n+1)\alpha\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}:$$

$$\sin\alpha + \sin 3\alpha + \sin 5\alpha + \&\text{c.} + \sin(2n-1)\alpha = \frac{\sin^2 n\alpha}{\sin\alpha}: \&\text{c.}$$

If $\alpha = \delta = 10^\circ$, we have, by the same formula,

$$\sin 10^\circ + \sin 20^\circ + \sin 30^\circ + \&\text{c.} + \sin 90^\circ = \frac{\sin 50^\circ}{\sqrt{2}\sin 5^\circ}.$$

72. Cor. The series, $\sin\alpha - \sin(\alpha\pm\delta) + \sin(\alpha\pm2\delta) - \&\text{c.}$ to n terms, may be summed in the same manner, by putting it into the form,

$$\{\sin\alpha + \sin(\alpha\pm2\delta) + \&\text{c.}\} - \{\sin(\alpha\pm\delta) + \sin(\alpha\pm3\delta) + \&\text{c.}\}.$$

73. To find the sum of n terms of the series,

$$\cos\alpha + \cos(\alpha+\delta) + \cos(\alpha+2\delta) + \&\text{c.}$$

Here, $2\sin\frac{1}{2}\delta\cos\alpha = \sin(\alpha+\frac{1}{2}\delta) - \sin(\alpha-\frac{1}{2}\delta)$:

$$2\sin\tfrac{1}{2}\delta\cos(\alpha+\delta) = \sin(\alpha+\tfrac{3}{2}\delta) - \sin(\alpha+\tfrac{1}{2}\delta):$$

$$2\sin\tfrac{1}{2}\delta\cos(\alpha+2\delta) = \sin(\alpha+\tfrac{5}{2}\delta) - \sin(\alpha+\tfrac{3}{2}\delta): \&\text{c.}:$$

$$2\sin\tfrac{1}{2}\delta\cos\{\alpha+(n-1)\delta\}$$

$$= \sin\{\alpha+\tfrac{1}{2}(2n-1)\delta\} - \sin\{\alpha+\tfrac{1}{2}(2n-3)\delta\}:$$

whence, we have $s = \dfrac{\cos\{\alpha+\frac{1}{2}(n-1)\delta\}\sin\frac{1}{2}n\delta}{\sin\frac{1}{2}\delta}$.

$$\operatorname{Cos} a + \cos 2a + \&\text{c.} + \cos na = \frac{\cos \frac{1}{2}(n+1)\,a \sin \frac{1}{2} na}{\sin \frac{1}{2} a}:$$

$$\cos a + \cos 3a + \cos 5a + \&\text{c.} + \cos (2n-1)\,a = \frac{\sin 2na}{2 \sin a}.$$

As none of the preceding series *converge* throughout their whole extent, their sums, when continued *in infinitum*, cannot be expressed by any general formulæ.

74. Cor. 1. Since, $2\sin^2 a = 1 - \cos 2a$, and $2\cos^2 a = 1 + \cos 2a$, it follows that the series,

$$\sin^2 a + \sin^2 (a+\delta) + \sin^2 (a + 2\delta) + \&\text{c. to } n \text{ terms},$$

$$\cos^2 a + \cos^2 (a+\delta) + \cos^2 (a + 2\delta) + \&\text{c. to } n \text{ terms},$$

may be summed by means of the formula above given.

Also, by successive *Differentiations* and *Integrations*, it will be easy to ascertain the sums of such series as,

$$\sin a + 2 \sin 2a + 3 \sin 3a + \&\text{c}:$$

$$\sin a + 2^2 \sin 2a + 3^2 \sin 3a + \&\text{c}: \ \&\text{c}:$$

$$\sin a + \tfrac{1}{2} \sin 2a + \tfrac{1}{3} \sin 3a + \&\text{c}:$$

$$\sin a + \frac{1}{2^2} \sin 2a + \frac{1}{3^2} \sin 3a + \&\text{c}: \ \&\text{c}.$$

75. Cor. 2. It will readily appear that the series, hitherto considered, are *recurring* series, and they may therefore be summed by means of the algebraical formulæ for that purpose: but if the formulæ of Articles (122) or (131) be used, the same object will be attained by the summation of *two* geometrical progressions.

76. To find the sum of n terms of a series of the form,

$$\sin a \cos\beta + \sin 2a \cos 3\beta + \&\text{c.} + \sin na \cos (2n-1)\beta.$$

Since, $2 \sin na \cos (2n-1)\,\beta$

$$= \sin \{na + (2n-1)\,\beta\} + \sin \{na - (2n-1)\,\beta\},$$

we have, $2 \sin a \cos \beta = \sin (a+\beta) + \sin (a - \beta)$:

$$2 \sin 2a \cos 3\beta = \sin (2a + 3\beta) + \sin (2a - 3\beta): \ \&\text{c.}:$$

and thus the series is resolved into two others, which belong to the preceding articles.

77. To find the sum of n terms of the series,

$$\operatorname{cosec} a + \operatorname{cosec} 2a + \operatorname{cosec} 2^2 a + \&\text{c.}$$

Here, $\operatorname{cosec} a = \cot \frac{1}{2}a - \cot a$:

$$\operatorname{cosec} 2a = \cot a - \cot 2a:$$

$$\operatorname{cosec} 2^2\alpha = \cot 2\alpha - \cot 2^2\alpha : \ \&c.:$$
$$\operatorname{cosec} 2^{n-1}\alpha = \cot 2^{n-2}\alpha - \cot 2^{n-1}\alpha :$$
$$\therefore\ s = \cot \tfrac{1}{2}\alpha - \cot 2^{n-1}\alpha.$$

78. To find the sum of n terms of the series,

$$\tan \alpha + 2 \tan 2\alpha + 2^2 \tan 2^2\alpha + \&c.$$

Here, $\tan \alpha = \cot \alpha - 2 \cot 2\alpha$:

$$2 \tan 2\alpha = 2 \cot 2\alpha - 2^2 \cot 2^2\alpha :$$
$$2^2 \tan 2^2\alpha = 2^2 \cot 2^2\alpha - 2^3 \cot 2^3\alpha : \ \&c.:$$
$$2^{n-1} \tan 2^{n-1} \alpha = 2^{n-1} \cot 2^{n-1}\alpha - 2^n \cot 2^n\alpha :$$
$$\therefore\ s = \cot \alpha - 2^n \cot 2^n\alpha.$$

By a similar process, the sum of n terms of the series,

$$\frac{1}{2} \tan \frac{1}{2}\,\alpha + \frac{1}{2^2} \tan \frac{1}{2^2}\,\alpha + \frac{1}{2^3} \tan \frac{1}{2^3}\,\alpha + \&c.,$$

s $\frac{1}{2^n} \cot \frac{1}{2^n}\alpha - \cot \alpha$; which, when $n = \infty$, becomes $\frac{1}{\alpha} - \cot \alpha$.

79. To find the sum of n terms of the series,

$$(\tan \alpha + \cot \alpha) + (\tan 2\alpha + \cot 2\alpha) + (\tan 3\alpha + \cot 3\alpha) + \&c.$$

Here, $\tan \alpha + \cot \alpha = 2 \cot \alpha - 2 \cot 2\alpha$:

$$\tan 2\alpha + \cot 2\alpha = 2 \cot 2\alpha - 2 \cot 2^2\alpha : \ \&c.:$$
$$\tan 2^{n-1}\alpha + \cot 2^{n-1}\alpha = 2 \cot 2^{n-1}\alpha - 2 \cot 2^n\alpha :$$
$$\therefore\ s = 2 \cot \alpha - 2 \cot 2^n\alpha.$$

80. To find the sum of n terms of the series,

$$\tan^{-1} \frac{1}{1+1+1^2} + \tan^{-1} \frac{1}{1+2+2^2} + \tan^{-1} \frac{1}{1+3+3^2} + \&c.$$

Since, $\tan^{-1} \frac{1}{n} - \tan^{-1} \frac{1}{n+1} = \tan^{-1} \frac{1}{1+n+n^2}$;

$$\therefore\ \tan^{-1} \frac{1}{1+1+1^2} = \tan^{-1} \frac{1}{1} - \tan^{-1} \frac{1}{2} :$$
$$\tan^{-1} \frac{1}{1+2+2^2} = \tan^{-1} \frac{1}{2} - \tan^{-1} \frac{1}{3} :$$
$$\tan^{-1} \frac{1}{1+3+3^2} = \tan^{-1} \frac{1}{3} - \tan^{-1} \frac{1}{4} : \ \&c.:$$
$$\tan^{-1} \frac{1}{1+n+n^2} = \tan^{-1} \frac{1}{n} - \tan^{-1} \frac{1}{n+1} :$$
$$\therefore\ s = \tan^{-1} 1 - \tan^{-1} \frac{1}{n+1} = \frac{1}{4}\pi - \tan^{-1} \frac{1}{n+1}.$$

81. To find the sum of n terms of the series,

$$x \sin \theta + x^2 \sin 2\theta + x^3 \sin 3\theta + \&c.$$

Here,

$$s = x \left\{ \frac{e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}}{2\sqrt{-1}} \right\} + x^2 \left\{ \frac{e^{2\theta\sqrt{-1}} - e^{-2\theta\sqrt{-1}}}{2\sqrt{-1}} \right\} + \&c.$$

$$= \frac{1}{2\sqrt{-1}} \left\{ xe^{\theta\sqrt{-1}} + x^2 e^{2\theta\sqrt{-1}} + x^3 e^{3\theta\sqrt{-1}} + \&c. \right\}$$

$$- \frac{1}{2\sqrt{-1}} \left\{ xe^{-\theta\sqrt{-1}} + x^2 e^{-2\theta\sqrt{-1}} + x^3 e^{-3\theta\sqrt{-1}} + \&c. \right\}$$

to n terms,

$$= \frac{xe^{\theta\sqrt{-1}}}{2\sqrt{-1}} \left\{ \frac{(xe^{\theta\sqrt{-1}})^n - 1}{xe^{\theta\sqrt{-1}} - 1} \right\} - \frac{xe^{-\theta\sqrt{-1}}}{2\sqrt{-1}} \left\{ \frac{(xe^{-\theta\sqrt{-1}})^n - 1}{xe^{-\theta\sqrt{-1}} - 1} \right\}$$

$$= \frac{1}{2\sqrt{-1}} \left\{ \frac{x^{n+2} (e^{n\theta\sqrt{-1}} - e^{-n\theta\sqrt{-1}})}{x^2 - (e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}})\, x + 1} \right\}$$

$$- \frac{1}{2\sqrt{-1}} \left\{ \frac{x^{n+1} (e^{(n+1)\theta\sqrt{-1}} - e^{-(n+1)\theta\sqrt{-1}})}{x^2 - (e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}})\, x + 1} \right\}$$

$$+ \frac{1}{2\sqrt{-1}} \left\{ \frac{x (e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}})}{x^2 - (e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}})\, x + 1} \right\}$$

$$= \frac{x^{n+2} \sin n\theta - x^{n+1} \sin (n+1)\theta + x \sin \theta}{x^2 - 2x \cos \theta + 1}.$$

If x be a proper fraction, and n be indefinitely great, we shall have

$$\sigma = \frac{x \sin \theta}{x^2 - 2x \cos \theta + 1}.$$

Nearly in the same manner, if s be the sum of n terms of $x \cos \theta + x^2 \cos 2\theta + x^3 \cos 3\theta + \&c.$, we find

$$s = \frac{x^{n+2} \cos n\theta - x^{n+1} \cos (n+1)\theta + x \cos \theta - x^2}{x^2 - 2x \cos \theta + 1}:$$

$$\text{and } \sigma = \frac{x \cos \theta - x^2}{x^2 - 2x \cos \theta + 1}.$$

If x be taken equal to ± 1, ± 2, ± 3, &c., the sums of the corresponding series will be obtained: and the same when the values of x are $\pm \frac{1}{\sqrt{2}}$, $\pm \frac{1}{\sqrt{3}}$, &c.

By means of the operations of *Differentiation* and *Integration* with respect to either x or θ, the sums of various other trigonometrical series may be determined: but the almost entire absence of utility renders it unnecessary to pursue the subject further at present.

82. To find the sum of all the straight lines, drawn from an angle of a regular polygon of n sides, inscribed n a circle, to each of the other angles.

Here, we have from the simplest principles,

$$s = r\{\text{chd}\,\theta + \text{chd}\,2\theta + \text{chd}\,3\theta + \&\text{c. to } n-1 \text{ terms}\}$$
$$= 2r\{\sin\alpha + \sin 2\alpha + \sin 3\alpha + \&\text{c. to } n-1 \text{ terms}\}:$$

$$\text{but in this case, } \theta = \frac{2\pi}{n}, \text{ and } \therefore\ \alpha = \frac{\pi}{n}:$$

$$\text{whence, } s = 2r\left\{\frac{\sin\frac{1}{2}n\alpha \sin\frac{1}{2}(n-1)\alpha}{\sin\frac{1}{2}\alpha}\right\} = 2r\cot\frac{\pi}{2n}.$$

It is evident also, that the sum of the squares of the said lines may be ascertained by means of the series,

$$\sin^2\alpha + \sin^2 2\alpha + \sin^2 3\alpha + \&\text{c. to } n-1 \text{ terms.}$$

83. To find the sum of the infinite series,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&\text{c.}$$

$$\text{Since, } a - \frac{a^3}{1.2.3} + \frac{a^5}{1.2.3.4.5} - \&\text{c. } \textit{in infinitum}$$

$$= \sin a = a\left(1 - \frac{a^2}{\pi^2}\right)\left(1 - \frac{a^2}{2^2\pi^2}\right) \&\text{c. } \textit{in infinitum}$$

$$= a - \frac{a^3}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&\text{c. } \textit{in infinitum}\right)$$

$$+ \frac{a^5}{\pi^4}\left(\frac{1}{1^2.2^2} + \frac{1}{1^2.3^2} + \frac{1}{2^2.3^2} + \&\text{c. } \textit{in infinitum}\right) - \&\text{c.};$$

we have, by equating the coefficients of a^3, in both sides,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&\text{c. } \textit{in infinitum} = \frac{\pi^2}{6}.$$

$$\text{Hence, also } \frac{1}{1^2.2^2} + \frac{1}{1^2.3^2} + \frac{1}{2^2.3^2} + \frac{1}{2^2.4^2} + \&\text{c.}$$

$$\textit{in infinitum} = \frac{\pi^4}{120}: \text{ and similarly of others.}$$

Precisely in the same manner, by means of the expressions for $\cos a$, it is found that

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\text{\&c. } \textit{in infinitum} = \frac{\pi^2}{8}:$$

$$\frac{1}{1^2.3^2}+\frac{1}{1^2.5^2}+\frac{1}{3^2.5^2}+\text{\&c. } \textit{in infinitum} = \frac{\pi^4}{384}.$$

84. We have seen in Article (160), that

$$\sin ra = 2^{r-1}\sin a \sin\left(a+\frac{\pi}{r}\right)\sin\left(a+\frac{2\pi}{r}\right)\text{ \&c.}$$

to r factors: whence, if $a = 0$, we have

$$r = 2^{r-1}\sin\frac{\pi}{r}\sin\frac{2\pi}{r}\sin\frac{3\pi}{r}\text{ \&c. to } r-1 \text{ factors}:$$

and thus any *whole number* whatever, may be expressed in the form of a continued product of trigonometrical functions combined with powers of the number 2.

By taking the logarithms of both members, and by the process of differentiation, we obtain

$$r\cot ra = \cot a + \cot\left(a+\frac{\pi}{r}\right)+\cot\left(a+\frac{2\pi}{r}\right)+\text{\&c. to } r \text{ terms.}$$

Repeating the latter process, we arrive at

$$r^2\operatorname{cosec}^2 ra = \operatorname{cosec}^2 a + \operatorname{cosec}^2\left(a+\frac{\pi}{r}\right)+\operatorname{cosec}^2\left(a+\frac{2\pi}{r}\right)$$
$$+\text{\&c. to } r \text{ terms}:$$

$$\text{whence, } r^2\operatorname{cosec}^2 ra - r = \cot^2 a + \cot^2\left(a+\frac{\pi}{r}\right)+\cot^2\left(a+\frac{2\pi}{r}\right)$$
$$+\text{\&c. to } r \text{ terms.}$$

Similar conclusions may be drawn from the like treatment of the corresponding expression for $\cos ra$: and it will not be difficult to shew that

$$\left\{\tan\frac{a}{r}+\tan\frac{2\pi+a}{r}+\tan\frac{4\pi+a}{r}+\text{\&c. to } r \text{ terms}\right\}$$
$$\times\left\{\cot\frac{a}{r}+\cot\frac{2\pi+a}{r}+\cot\frac{4\pi+a}{r}+\text{\&c. to } r \text{ terms}\right\}$$

$= r^2$, if r be any whole number not divisible by 4.

XI. *Various Theorems and Observations.*

85. If $\cos\alpha = \frac{1}{2}\left(x + \frac{1}{x}\right)$, and $\therefore \sin\alpha = \frac{1}{2\sqrt{-1}}\left(x - \frac{1}{x}\right)$: also, if $\cos\beta = \frac{1}{2}\left(y + \frac{1}{y}\right)$, and $\therefore \sin\beta = \frac{1}{2\sqrt{-1}}\left(y - \frac{1}{y}\right)$: by substitution and reduction, we shall obtain

$$\cos(\alpha+\beta) = \tfrac{1}{2}\left(xy + \frac{1}{xy}\right), \text{ and } \cos(\alpha-\beta) = \tfrac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right):$$

$$\sin(\alpha+\beta) = \frac{1}{2\sqrt{-1}}\left(xy - \frac{1}{xy}\right), \text{ and } \sin(\alpha-\beta) = \frac{1}{2\sqrt{-1}}\left(\frac{x}{y} - \frac{y}{x}\right)$$

Also, if $\cos\gamma = \frac{1}{2}\left(z + \frac{1}{z}\right)$, and $\therefore \sin\gamma = \frac{1}{2\sqrt{-1}}\left(z - \frac{1}{z}\right)$,

$$\text{we find } \cos(\alpha+\beta+\gamma) = \tfrac{1}{2}\left(xyz + \frac{1}{xyz}\right),$$

$$\text{and } \sin(\alpha+\beta+\gamma) = \frac{1}{2\sqrt{-1}}\left(xyz - \frac{1}{xyz}\right):$$

and thence by induction, *whatever* be the number of angles, we shall have

$$\cos(\alpha+\beta+\gamma+\&c.) = \tfrac{1}{2}\left(xyz\,\&c. + \frac{1}{xyz\,\&c.}\right),$$

$$\sin(\alpha+\beta+\gamma+\&c.) = \frac{1}{2\sqrt{-1}}\left(xyz\,\&c. - \frac{1}{xyz\,\&c.}\right).$$

Results *analogous* to these will be obtained, when one or more of the angles are *negative.*

86. Hence, by the ordinary processes, we shall have

$$\cos(\alpha+\beta+\gamma+\&c.) + \sqrt{-1}\sin(\alpha+\beta+\gamma+\&c.) = xyz\,\&c.:$$

$$\cos(\alpha+\beta+\gamma+\&c.) - \sqrt{-1}\sin(\alpha+\beta+\gamma+\&c.) = \frac{1}{xyz\,\&c.}:$$

and if $\alpha = \beta = \gamma = \&c.$, and $\therefore x = y = z = \&c.$, we recognize the results of Article (122).

The simplest considerations will now conduct us to the Theorems established in Article (63), and also to the *Proof* of *Demoivre's* Theorem, when the index is a *positive whole number.*

87. Retaining the same notation, we shall have

$$2\cos(m\alpha+n\beta)=x^m y^n+\frac{1}{x^m y^n},\ 2\cos(m\alpha-n\beta)=\frac{x^m}{y^n}+\frac{y^n}{x^m}:$$

$$2\sqrt{-1}\sin(m\alpha+n\beta)=x^m y^n-\frac{1}{x^m y^n},$$

$$2\sqrt{-1}\sin(m\alpha-n\beta)=\frac{x^m}{y^n}-\frac{y^n}{x^m}.$$

88. When the index is a positive integer, it appears above that the process employed in establishing *Demoivre's* Theorem is the same as that generally used to prove *Newton's* Binomial Theorem: and their results are the same except so far as $(a+b)^m$ is influenced by the relation $b=\sqrt{-1}\sqrt{a^2-1}$, which produces in them a modification never contemplated in *Arithmetic.*

Thus, from $\cos m\theta=\cos^m\theta-\frac{m(m-1)}{1.2}\cos^{m-2}\theta\sin^2\theta+\&c.,$

and $\sin m\theta=m\cos^{m-1}\theta\sin\theta-\frac{m(m-1)(m-2)}{1.2.3}\cos^{m-3}\theta\sin^3\theta+\&c.,$

if we suppose $\theta=45^\circ$, and $\therefore\ \cos\theta=\sin\theta=\frac{1}{\sqrt{2}}$, we have

$$2^{\frac{m}{2}}\cos\left(\frac{m\pi}{4}\right)=1-\frac{m(m-1)}{1.2}+\frac{m(m-1)(m-2)(m-3)}{1.2.3.4}-\&c.,$$

$$\text{and } 2^{\frac{m}{2}}\sin\left(\frac{m\pi}{4}\right)=m-\frac{m(m-1)(m-2)}{1.2.3}+\&c.,$$

which are *analogous* to the sums of the coefficients of the *odd* and *even* terms in the expansion of $(1\pm v)^m$, as found in the *Algebra*, where m is a general symbol.

Also, $(a+b)^{\frac{m}{n}}=\{a+\sqrt{-1}\sqrt{a^2-1}\}^{\frac{m}{n}}$ has n different values, which, though not immediately apparent by the binomial expansion, are at once made manifest by means of *Demoivre's* Theorem.

89. It is seen immediately that $\theta=\cos^{-1}\frac{1}{2}\left(x+\frac{1}{x}\right)$ $=\sin^{-1}\frac{1}{2\sqrt{-1}}\left(x-\frac{1}{x}\right)$ is *imaginary*, when x is any *real* quantity whatever $>$ or <1, because $\frac{1}{2}\left(x+\frac{1}{x}\right)$ is necessarily >1: but the correctness of the conclusions we have

rived at *solely* through the intervention of *imaginary* *antities* being easily verified when m is a positive hole number, the principles of *Symbolical Algebra* leave in no doubt as to their applicability whenever they n be rendered available for *approximate* arithmetical mputations.

90. The expressions $\cos\theta \pm \sqrt{-1}\sin\theta$, as used in e *fifth* chapter, involve an *imaginary angle* θ, though e have hitherto regarded $\cos\theta$ and $\sin\theta$ as symbols of *antity*, and treated them according to the rules of *rithmetical Algebra*, with the additional consideration the effects of the symbol $\sqrt{-1}$.

As *instruments* of Analysis, they have already proved be of great utility, by enabling us immediately to mpare the quantities *unaffected* and *affected by* $\sqrt{-1}$, the *two* members of an *equality*: and it may now be quired how far a *real angle* θ may be applied to plain the *Geometrical* meanings of the *Algebraical* pressions $a \pm b\sqrt{-1}$, which have been treated of in rticle (121).

In the diagram of Article (28), let A be joined with e points P: then, if $AM = a$, $MP = b$, and $\angle MAP = \theta$, e have $AP = \sqrt{a^2 + b^2}$; and also,

$$\cos\theta = \frac{a}{\sqrt{a^2 + b^2}},\ \sin\theta = \frac{b}{\sqrt{a^2 + b^2}},\ \text{and } \tan\theta = \frac{b}{a}:$$

whence, $\sqrt{a^2 + b^2}\,\{\cos\theta + \sqrt{-1}\sin\theta\} = a + b\sqrt{-1}$:

at is, $\sqrt{a^2 + b^2}$ the *arithmetical* magnitude of the line P being *affected* according to the rule of signs in *gebra*, with the quantity $\cos\theta \pm \sqrt{-1}\sin\theta$, which inlves its *geometrical position*, leads us to conclude that if P be represented by $\sqrt{a^2 + b^2}$, when *unaffected*, or ing in the direction denoted by the sign +, it will be equately represented both in *magnitude* and *position* $\sqrt{a^2 + b^2}\,\{\cos\theta + \sqrt{-1}\sin\theta\}$, when it makes an $\angle\theta$ th the *primitive conventional* direction.

Hence, under such circumstances, we may regard s $\theta \pm \sqrt{-1}\sin\theta$ merely as *signs* of *affection* in the me way as + *and* − *have* been received in Article (28): *us, if*

$\theta = 0^\circ$, we have $b = 0$, and $\therefore AP = a$:
$\theta = 180^\circ$, ... $b = 0$, and $\therefore AP = -a$:
$\theta = 90^\circ$, ... $a = 0$, and $\therefore AP = b\sqrt{-1}$:
$\theta = 270^\circ$, ... $a = 0$, and $\therefore AP = -b\sqrt{-1}$:

and these agree in all respects with what has been laid down in the same Article, and in the Appendix to the *Author's* Algebra.

91. Hence the *symbolical* representations of the line AP whose *arithmetical* magnitude is $\sqrt{a^2 + b^2}$, will be

in the *first* quadrant, $AP = a + b\sqrt{-1}$:
.... *second* ...., $AP = -a + b\sqrt{-1}$:
.... *third* ...., $AP = -a - b\sqrt{-1}$:
.... *fourth* ...., $AP = a - b\sqrt{-1}$.

Also, the area of the $\triangle\, PAP$ formed in the *first* and *fourth* quadrants will be

$$= \tfrac{1}{2} AP \,.\, AP \sin\angle PAP$$
$$= \tfrac{1}{2}(a + b\sqrt{-1})(a - b\sqrt{-1})\, 2 \sin\theta \cos\theta$$
$$= \tfrac{1}{2}(a^2 + b^2)\frac{2ab}{a^2 + b^2} = ab = AM \,.\, MP,$$

as we know to be the case from common considerations.

92. It now appears that every line in plane Geometry, when expressed so as to point out its *position* as well as its *magnitude*, may be represented by a quantity of the *form* $a + b\sqrt{-1} = (\cos\theta + \sqrt{-1}\sin\theta)\sqrt{a^2 + b^2}$, the *magnitude* being $= \sqrt{a^2 + b^2}$, and the *position* being decided by the *sign* $\cos\theta + \sqrt{-1}\sin\theta$, where $\theta = \tan^{-1}\frac{b}{a}$.

On these grounds, the n roots of the equation $x^n - a^n \pm 0$, whose solution is found in Article (153), to be

$$x = \left\{\cos\frac{2r\pi}{n} + \sqrt{-1}\sin\frac{2r\pi}{n}\right\} a,$$

will denote n straight lines each equal in *length* to a which are *inclined* to the *primitive* direction at angles determined by assigning to r, the values 0, 1, 2, 3, &c., $n - 1$.

These views and interpretations of $\cos\theta + \sqrt{-1}\sin\theta$

considered as a *Sign* of *Affection*, seem to have been suggested by *Mr Gompertz*, President of the Mathematical Society in London: they have been followed out by *Mr Warren* in his *Treatise on the Geometrical Representation of the Square Roots of Negative Quantities:* and the student will find them amply developed in Professor *Peacock's* Algebra. They lead to the treatment of *Imaginary Angles*, as well as to the consideration of *Imaginary Logarithms*, which want of room here compels us to omit. See also the *Cambridge Mathematical Journal.*

93. The base e of the *Napierian* Logarithms is an incommensurable quantity.

For, if possible, let the rational fraction $\frac{m}{n} = e$

$$= 2 + \frac{1}{1.2} + \frac{1}{1.2.3} + \&c. + \frac{1}{1.2.3.\&c.\,n} + \frac{1}{1.2.3.\&c.\,(n+1)} + \&c.:$$

whence, multiplying both sides by $1.2.3.\&c.\;n$, we have

$$1.2.3.\&c.(n-1)m = 2.2.3.4.\&c.\,n + 3.4.5.\&c. + \&c.$$

$$+ \left\{\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \&c.\right\}:$$

but the former member being integral, the latter must be so too, which is impossible since the last line is less than

$$\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \&c. \textit{ in inf.} < \frac{1}{n}:$$

hat is, e cannot be expressed by a *rational* magnitude, and is therefore *incommensurable.*

Also, the error incurred by stopping after n terms

$$= \frac{1}{1.2.3.\&c.\,n}\left\{\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \&c. \textit{ in inf.}\right\}$$

$$< \frac{1}{1.2.3.\&c.\,n}\left\{\frac{1}{n}\right\} < \frac{1}{1.2.3.\&c.\,(n-1)n^2}.$$

94. Since $\log_e(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \&c.:$

$$\text{and } \log_e\left(1 + \frac{1}{z}\right) = \frac{1}{z} - \frac{1}{2z^2} + \frac{1}{3z^3} - \&c.:$$

we shall have $\log_e z = \log_e(1+z) - \log_e\left(1 + \frac{1}{z}\right)$

$$= \left(z - \frac{1}{z}\right) - \frac{1}{2}\left(z^2 - \frac{1}{z^2}\right) + \frac{1}{3}\left(z^3 - \frac{1}{z^3}\right) - \&c.:$$

an equality which can be *arithmetically* true only when $z = 1$: but if we suppose $z = \sqrt{-1}$, we shall have

$$\log_e \sqrt{-1} = 2\sqrt{-1}\,\{1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \&c.\} = 2\sqrt{-1}\,(\tfrac{1}{4}\pi),$$

as appears from Article (135):

$$\text{whence, } \pi = \frac{2 \log_e \sqrt{-1}}{\sqrt{-1}}:$$

or, 3.1415926 &c. is expressed by *imaginary quantities*, consistently with the character of *Symbolical Algebra*.

Also, from $z^x = e^{x \log_e z}$, we shall have

$$(\sqrt{-1})^{\sqrt{-1}} = e^{\sqrt{-1} \log_e \sqrt{-1}} = e^{-\frac{\pi}{2}}$$

$$= 1 - \frac{\pi}{2} + \frac{1}{1.2}\,\frac{\pi^2}{4} - \frac{1}{1.2.3}\,\frac{\pi^3}{8} + \&c. = .207879\ \&c.$$

The formulæ are the discoveries of *John Bernoulli*.

95. If in the series for $\log_e z$, we put $z = e^{\theta\sqrt{-1}}$, we shall have

$$\theta\sqrt{-1} = (e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}) - \tfrac{1}{2}(e^{2\theta\sqrt{-1}} - e^{-2\theta\sqrt{-1}})$$
$$+ \tfrac{1}{3}(e^{3\theta\sqrt{-1}} - e^{-3\theta\sqrt{-1}}) - \&c.:$$

from which is obtained by means of Article (131),

$$\tfrac{1}{2}\theta = \sin\theta - \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta - \&c.\ \textit{in inf.}:$$

and to this the remark above may be applied.

The converse of this may be established by finding the sum of the more general series,

$$a\sin\theta - \tfrac{1}{2}a^2\sin 2\theta + \tfrac{1}{3}a^3\sin 3\theta - \&c.\ \textit{in inf.}$$

For, using the imaginary exponential forms, we have

$$2\sqrt{-1}\,\sigma = a\left(x - \frac{1}{x}\right) - \tfrac{1}{2}a^2\left(x^2 - \frac{1}{x^2}\right) + \tfrac{1}{3}a^3\left(x^3 - \frac{1}{x^3}\right) - \&c.$$

$$= (ax - \tfrac{1}{2}a^2x^2 + \tfrac{1}{3}a^3x^3 - \&c.) - \left(\frac{a}{x} - \frac{1}{2}\frac{a^2}{x^2} + \frac{1}{3}\frac{a^3}{x^3} - \&c.\right)$$

$$= \log_e(1 + ax) - \log_e\left(1 + \frac{a}{x}\right) = \log_e\left(\frac{1 + ax}{1 + ax^{-1}}\right):$$

$$\text{whence, } e^{2\sqrt{-1}\,\sigma} = \frac{1 + ax}{1 + ax^{-1}}, \text{ and } \therefore\ \frac{e^{2\sqrt{-1}\,\sigma} - 1}{e^{2\sqrt{-1}\,\sigma} + 1}$$

$$= \frac{a(x - x^{-1})}{2 + a(x + x^{-1})} = \frac{2a\sqrt{-1}\sin\theta}{2 + 2a\cos\theta}: \text{ or, } \tan\sigma = \frac{a\sin\theta}{1 + a\cos\theta}.$$

If $a = 1$, we have $\sigma = \tan^{-1} \dfrac{\sin\theta}{1 + \cos\theta} = \tan^{-1} \tan \tfrac{1}{2}\theta = \tfrac{1}{2}\theta$.

If $a = h \sin\theta$, we have $\sigma = \tan^{-1} \dfrac{h}{1 + h \cot\theta + \cot^2\theta}$,

which, if $t = \cot\theta$, gives $\tan^{-1}(t + h) - \tan^{-1} t$

$= h \sin\theta \sin\theta - \tfrac{1}{2} h^2 \sin^2\theta \sin 2\theta + \text{\&c.}$ *in inf.*:

and by assigning to h different values, many curious analytical expressions may be deduced.

In a similar manner, it may be proved that

$$1 + \frac{a}{1}\cos\theta + \frac{a^2}{1.2}\cos 2\theta + \frac{a^3}{1.2.3}\cos 3\theta + \text{\&c. } in\ inf.$$

$$= \tfrac{1}{2}\left(e^{ax} + e^{\frac{a}{x}}\right) = e^{a \cos\theta} \cos(a \sin\theta).$$

96. Since $\sin\theta = 2 \sin\tfrac{1}{2}\theta \cos\tfrac{1}{2}\theta$:

$$\sin\tfrac{1}{2}\theta = 2 \sin\frac{1}{2^2}\theta \cos\frac{1}{2^2}\theta:$$

$$\sin\frac{1}{2^2}\theta = 2 \sin\frac{1}{2^3}\theta \cos\frac{1}{2^3}\theta: \text{\&c.}: \sin\frac{1}{2^{n-1}}\theta = 2 \sin\frac{1}{2^n}\theta \cos\frac{1}{2^n}\theta:$$

by multiplication of the corresponding members, we have

$$\sin\theta = 2^n \cos\tfrac{1}{2}\theta \cos\frac{1}{2^2}\theta \cos\frac{1}{2^3}\theta \text{ \&c. } \sin\frac{1}{2^n}\theta \cos\frac{1}{2^n}\theta:$$

$$\therefore\ 2^n \sin\frac{1}{2^n}\theta = \sin\theta \sec\tfrac{1}{2}\theta \sec\frac{1}{2^2}\theta \sec\frac{1}{2^3}\theta \text{ \&c. } \sec\frac{1}{2^n}\theta,$$

and this, by Article (86), when n is *infinite*, gives

$$\theta = \sin\theta \sec\tfrac{1}{2}\theta \sec\tfrac{1}{4}\theta \sec\tfrac{1}{8}\theta \text{ \&c. } in\ infinitum,$$

which is known by the name of *Euler's* Formula.

97. It has been proved in Article (81. *Ap.*) that

$$\sin\theta + x \sin 2\theta + x^2 \sin 3\theta + \text{\&c. } in\ inf. = \frac{\sin\theta}{1 - x(2\cos\theta - x)}$$

$$= \sin\theta \left\{1 + x(2\cos\theta - x) + \text{\&c.} + x^{n-1}(2\cos\theta - x)^{n-1} + \text{\&c.}\right\}:$$

whence, by equating the coefficients of x^{n-1} after expansion,

we have $\dfrac{\sin n\theta}{\sin\theta} = (2\cos\theta)^{n-1} - \dfrac{n-2}{1}(2\cos\theta)^{n-3}$

$$+ \frac{(n-3)(n-4)}{1.2}(2\cos\theta)^{n-5} - \text{\&c.}$$

In the same way, from the sum of the series

$$\cos\theta + x\cos 2\theta + x^2\cos 3\theta + \&c. \textit{ in inf.} = \frac{\cos\theta - x}{1 - x(2\cos\theta - x)},$$

it may be shewn that

$$\frac{\cos n\theta}{\cos\theta} = (2\cos\theta)^{n-1} - \frac{n}{1}(2\cos\theta)^{n-3} + \frac{n(n-3)}{1.2}(2\cos\theta)^{n-5} - \&c.:$$

and it is not difficult by means of their *general terms*, to transform the values of $\sin n\theta$ and $\cos n\theta$ thus found, so as to comprise sines or cosines *only*, though our space will not allow of their insertion here.

A system of Trigonometry depending upon the *Area* of a *Hyperbolic Sector*, has been drawn up, and as it satisfactorily explains several results in *Analysis* which appear to involve inconsistencies, it will be the *Author's* endeavour to publish it in some *other* work, because the extent of this has already exceeded his expectations and wishes.

CHAPTER II.

MISCELLANEOUS THEOREMS AND PROBLEMS FOR PRACTICE.

I. *Functions of Simple Angles.*

1. THE complement of $45^\circ \pm A$ is $45^\circ \mp A$: and the complement of $90^\circ \pm A$ is $\mp A$.

2. The supplement of $90^\circ \pm A$ is $90^\circ \mp A$: and the supplement of $180^\circ \pm A$ is $\mp A$.

3. The difference of an angle and its complement, is equal to the complement of twice the angle.

4. The supplement of an angle is equal to twice the complement of half the angle.

5. If Δ and δ be the magnitudes of an English and Foreign degree, then will $\Delta = \delta + \frac{1}{9}\delta$, and $\delta = \Delta - \frac{1}{10}\Delta$.

6. Prove that the ratios of the lengths of a Foreign and English degree, minute and second, are expressed by the fractions $\frac{3 . 3}{2 . 5}$, $\frac{3 . 3^2}{2 . 5^2}$ and $\frac{3 . 3^3}{2 . 5^3}$, respectively.

7. Prove that $m' = m^{`} + \frac{23}{27} m^{`}$: and $n'' = 3n'' + \frac{7}{81} n''$.

8. The sum of two angles in the English scale is 75°, and their difference in the Foreign scale is $26\frac{1}{3}^{g}$: express them in both scales.

Answers: $49^{\circ}\,21'$, $25^{\circ}\,39'$, and $54\frac{5}{6}^{g}$, $28\frac{1}{2}^{g}$.

9. Shew that the circular measures of 30° and 45° are .523598 &c. and .785398 &c., respectively.

10. The angles, whose circular measures are $\frac{1}{2}$ and $\frac{3}{4}$, are $28^{\circ}\,38'\,52''\,24'''$ &c., and $42^{\circ}.9718346$ &c.

11. Prove the truth of the following expressions:

$$\sin a = \cos a \tan a = \frac{\cos a}{\cot a} = \frac{\tan a}{\sec a} = \frac{\cos a \sec a}{\operatorname{cosec} a} = \frac{\tan a \cot a}{\operatorname{cosec} a}:$$

$$\cos a = \sin a \cot a = \frac{\sin a}{\tan a} = \frac{\cot a}{\operatorname{cosec} a} = \frac{\sin a \operatorname{cosec} a}{\sec a} = \frac{\tan a \cot a}{\sec a}:$$

$$\operatorname{vers} a = \frac{\tan a - \sin a}{\tan a} = \frac{\operatorname{cosec} a - \cot a}{\operatorname{cosec} a} = \frac{\sec a - 1}{\sec a}:$$

$$\tan a = \frac{\sec a}{\operatorname{cosec} a} = \frac{\sin a \operatorname{cosec} a}{\cot a} = \frac{\cos a \sec a}{\cot a} = \frac{\cos a}{\sin a \cot^2 a}:$$

$$\cot a = \frac{\operatorname{cosec} a}{\sec a} = \frac{\sin a \operatorname{cosec} a}{\tan a} = \frac{\cos a \sec a}{\tan a} = \frac{\sin a}{\cos a \tan^2 a}:$$

$$\sec a = \frac{\tan a}{\sin a} = \frac{\operatorname{cosec} a}{\cot a} = \frac{\sin a \operatorname{cosec} a}{\cos a} = \frac{\tan a \cot a}{\cos a}:$$

$$\operatorname{cosec} a = \frac{\cot a}{\cos a} = \frac{\sec a}{\tan a} = \frac{\cos a \sec a}{\sin a} = \frac{\tan a \cot a}{\sin a}.$$

12. Shew that $\operatorname{vers}(180^{\circ} - a) = 2 - \operatorname{vers} a = 1 + \cos a$:

and $\operatorname{chd}^2(180^{\circ} - a) = 4 - \operatorname{chd}^2 a$.

13. Prove that $\operatorname{chd}^2 a = \operatorname{vers}^2 a + \sin^2 a = 2 - 2\cos a$: $\operatorname{chd}^2(180^{\circ} - a) = 2 + 2\cos a$: and $\operatorname{chd}^2(90^{\circ} - a) = 2 - 2\sin a$.

14. Prove that $\sec^2 a \operatorname{cosec}^2 a = \sec^2 a + \operatorname{cosec}^2 a$: and $\cos^2 a \cot^2 a = \cot^2 a - \cos^2 a$: also, $\sin^2 a \tan^2 a = \tan^2 a - \sin^2 a$.

15. If S and s be the sines of two angles; C and c their cosines: then will

$$(S^2 - S^2 s^2)(C^2 - C^2 c^2) = (s^2 - S^2 s^2)(c^2 - C^2 c^2).$$

16. If S and s be the secants of two angles; T and t their tangents: then will

$$S^2 T^2 (s^2 - 1)(t^2 + 1) = s^2 t^2 (S^2 - 1)(T^2 + 1).$$

17. Prove that $\sin^2\theta\cos^2\phi - \cos^2\theta\sin^2\phi = \sin^2\theta - \sin^2\phi$: and $\cos^2\theta\cos^2\phi - \sin^2\theta\sin^2\phi = \cos^2\theta - \sin^2\phi$.

18. Shew that $\dfrac{\cos^2\phi - \cos^2\theta}{\cos^2\theta\cos^2\phi} = \tan^2\theta - \tan^2\phi$:

and $\dfrac{\cos^2\theta - \sin^2\phi}{\sin^2\theta\sin^2\phi} = \cot^2\theta\cot^2\phi - 1 = \dfrac{1 - \tan^2\theta\tan^2\phi}{(\tan\theta\tan\phi)^2}$.

19. If S and s be the sines or secants of two angles, T and t their tangents: then will $\dfrac{T}{t}$ be greater or less than $\dfrac{S}{s}$, according as T is greater or less than t.

20. If C and c be the cosines or cosecants of two angles, T and t their cotangents: then is $\dfrac{C}{c}$ greater or less than $\dfrac{T}{t}$, according as T is greater or less than t.

21. If $\tan\alpha = 2\dfrac{\sin\alpha}{\cos\alpha}$ to the radius 2, it is required to prove that $\tan\alpha = 3\dfrac{\sin\alpha}{\cos\alpha}$, to the radius 3.

22. If $\sec^2\alpha = 16 + \tan^2\alpha$, to the radius 4: then will $\sec^2\alpha = 100 + \tan^2\alpha$, to the radius 10.

23. Prove that $\dfrac{\tan\alpha + \tan\beta}{\cot\alpha + \cot\beta} = \dfrac{\tan\alpha\tan\beta}{r^2}$, where the radius of the arcs is r.

24. Shew that $\text{chd}^2(90^\circ + \alpha)\,\text{chd}^2(90^\circ - \alpha)$ $= 4r^2\,\text{vers}(90^\circ + \alpha)\,\text{vers}(90^\circ - \alpha)$, to the radius r.

25. If $\sin\theta + \cos\phi = m$, and $\cos\theta + \sin\phi = n$: prove that $2m\sin\theta + 2n\cos\theta = m^2 + n^2 = 2m\cos\phi + 2n\sin\phi$.

26. If $\sec\theta + \sec\phi = p$, and $\tan\theta + \tan\phi = q$: prove that $2p\sec\theta - 2q\tan\theta = p^2 - q^2 = 2p\sec\phi - 2q\tan\phi$.

27. If $a^2\sin\theta\sin\phi + b^2\cos\theta\cos\phi = 0$; prove that $\sin^2\phi = \dfrac{b^4\cos^2\theta}{a^4\sin^2\theta + b^4\cos^2\theta}$, and $\cos^2\theta = \dfrac{a^4\sin^2\phi}{a^4\sin^2\phi + b^4\cos^2\phi}$.

28. If l be the length of an arc of 45° of a circle whose radius is r, and L be the length of an arc of 60°, of one whose radius is R: prove that $3rL = 4Rl$.

29. If $\sin\theta = p$, and $\tan\theta = q$: find $\tan\phi$, when $\operatorname{cosec}\phi = s$, the radius being unknown.

Answer: $$\text{Tan}\,\phi = \frac{p^2q^2}{\sqrt{(s^2+q^2)\,p^4 - 2p^2q^2s^2 + (s^2-p^2)\,q^4}}.$$

II. *Functions of Compound Angles.*

1. $$\text{Sin}\,a = \frac{2\sin\frac{1}{2}a}{\sec\frac{1}{2}a} = \frac{2\cos\frac{1}{2}a}{\operatorname{cosec}\frac{1}{2}a} = \frac{2\sin^2\frac{1}{2}a}{\tan\frac{1}{2}a}$$
$$= \frac{2\cos^2\frac{1}{2}a}{\cot\frac{1}{2}a} = \frac{2\tan\frac{1}{2}a}{\sec^2\frac{1}{2}a} = \frac{2\cot\frac{1}{2}a}{\operatorname{cosec}^2\frac{1}{2}a} = \frac{2}{\tan\frac{1}{2}a + \cot\frac{1}{2}a}$$
$$= \frac{1}{\tan\frac{1}{2}a + \cot a} = \frac{1}{\cot\frac{1}{2}a - \cot a}.$$

2. $$\text{Cos}\,a = \cos^4\tfrac{1}{2}a - \sin^4\tfrac{1}{2}a = \frac{2\cos\frac{1}{2}a - \sec\frac{1}{2}a}{\sec\frac{1}{2}a}$$
$$= \frac{\operatorname{cosec}\frac{1}{2}a - 2\sin\frac{1}{2}a}{\operatorname{cosec}\frac{1}{2}a} = \frac{\cot\frac{1}{2}a - \tan\frac{1}{2}a}{\cot\frac{1}{2}a + \tan\frac{1}{2}a} = \sin a\cot\tfrac{1}{2}a - 1$$
$$= \frac{1}{1 + \tan a\tan\frac{1}{2}a}.$$

3. $$\text{Tan}\,a = \frac{2\tan\frac{1}{2}a}{2 - \sec^2\frac{1}{2}a} = \frac{2}{\cot\frac{1}{2}a - \tan\frac{1}{2}a} = \frac{2\cot\frac{1}{2}a}{\cot^2\frac{1}{2}a - 1}$$
$$= \frac{2 + 2\tan a\tan\frac{1}{2}a}{\tan\frac{1}{2}a + \cot\frac{1}{2}a} = (1+\sec a)\tan\tfrac{1}{2}a = \tfrac{1}{2}\sec^2\tfrac{1}{2}a\,(\sin a + \tan a)$$
$$= \frac{1 - \sec a + (1 + \sec a)\tan\frac{1}{2}a}{1 - \tan\frac{1}{2}a}.$$

4. $$\text{Sin}\,a = 2\,\frac{\operatorname{vers}\frac{1}{2}a - \operatorname{covers}\frac{1}{2}a}{\sec\frac{1}{2}a - \operatorname{cosec}\frac{1}{2}a} = 2\sqrt{\frac{\sin^2\frac{1}{2}a - \cos^2\frac{1}{2}a}{\tan^2\frac{1}{2}a - \cot^2\frac{1}{2}a}}.$$

5. $$\text{Tan}\,\tfrac{1}{2}a = \frac{\cos a}{1+\cos a} \times \frac{\sin 2a}{1+\cos 2a} = \pm\sqrt{\frac{2\sin a - \sin 2a}{2\sin a + \sin 2a}}.$$

6. $\text{Sin}\,a\sin 3a = \sin^2 2a - \sin^2 a$, and $\sin a\sin 5a = \sin^2 3a - \sin^2 2a$: also, $\cos 3a\cos 5a = \cos^2 4a - \sin^2 a$.

7. $\text{Tan}\,\frac{1}{2}a\tan\frac{3}{2}a = \dfrac{\cos a - \cos 2a}{\cos a + \cos 2a}$, and $\tan a\tan 3a = \dfrac{\tan^2 2a - \tan^2 a}{1 - \tan^2 a\tan^2 2a}$: also, $\operatorname{chd}\frac{3}{2}a\operatorname{chd}\frac{1}{2}a = \operatorname{chd}^2 a - \operatorname{chd}^2\frac{1}{2}a$.

8. $$\text{Tan}\,2a - \tan a = \frac{\sin a}{2\cos^3 a - \cos a} = \frac{2\sin a}{\cos 3a + \cos a}.$$

9. $\mathrm{Cos}^2 2a - \sin^2 a = \cos a \cos 3a$, and $\tan^2 2a - \tan^2 a$
$= \dfrac{\sin a \sin 3a}{\cos^2 a \cos^2 2a}$: also, $\mathrm{cosec}^2 2a - \cot^2 a = \dfrac{1 - 4\cos^4 a}{\sin^2 2a}$.

10. $\mathrm{Tan}^4 a = \dfrac{\sin^2 2a - 4\sin^2 a}{\sin^2 2a + 4\sin^2 a - 4} = \dfrac{\cos^2 2a - 4\cos^2 a + 3}{\cos^2 2a + 4\cos^2 a - 1}$.

11. $\dfrac{\mathrm{Sin}\, 2a + \sin a}{\cos 2a + \cos a} = \tan \tfrac{3}{2}a$, $\dfrac{\sin 2a - \sin a}{\cos 2a - \cos a} = -\cot \tfrac{3}{2}a$,

and $\cot^2 (45^\circ + \tfrac{1}{2}a) = \dfrac{2\,\mathrm{cosec}\, 2a - \sec a}{2\,\mathrm{cosec}\, 2a + \sec a}$.

12. $\dfrac{\mathrm{Sec}\, a \sec 2a}{\sec 3a} + \sec 2a = 2$: $\dfrac{1 + \sin a}{1 + \cos a} = \tfrac{1}{2}(1 + \tan \tfrac{1}{2}a)^2$.

13. $\left(1 - \dfrac{2\tan a}{\tan 2a}\right)^{\frac{1}{2}} = \dfrac{2}{\cot \frac{1}{2}a - \tan \frac{1}{2}a}$, $\dfrac{1 - \sin 2a}{(1 - \tan a)^2} = \cos^2 a$.

14. $\dfrac{\mathrm{Cos}\, 3a}{\cos a} = \dfrac{1 - \tan a \tan 2a}{1 + \tan a \tan 2a}$, and $\dfrac{\tan a + \sec a}{\cot a + \mathrm{cosec}\, a}$
$= \tan \tfrac{1}{2}a \tan \left(45^\circ + \dfrac{a}{2}\right)$: also, $\dfrac{\mathrm{cosec}\, a - 2\,\mathrm{cosec}\, 3a}{\sec a} = \cot 3a$.

15. $\mathrm{Tan}\, a = \dfrac{\sin \frac{2}{3}a + \sin \frac{4}{3}a}{\cos \frac{2}{3}a + \cos \frac{4}{3}a}$, $\tan 2a = \dfrac{\sin a + \sin 3a}{\cos a + \cos 3a}$,

and $\tan 3a = \dfrac{\sin a + \sin 3a + \sin 5a}{\cos a + \cos 3a + \cos 5a}$.

16. $\dfrac{\sqrt{1 - \cos 2a}}{\sin 2a} = \dfrac{1}{\sqrt{2}} \sec a$, and $\sqrt{\dfrac{1 + \sin 2a}{\sin 2a}}$
$= \sqrt{\tfrac{1}{2}\tan a} + \sqrt{\tfrac{1}{2}\cot a}$: also, $\tan 3a - \tan a = 2\sec 3a \sin a$.

17. $\left.\begin{array}{l} 2\sin(45^\circ \pm a) = \sqrt{1 + \cos 2a} \pm \sqrt{1 - \cos 2a} \\ 2\cos(45^\circ \pm a) = \sqrt{1 + \cos 2a} \mp \sqrt{1 - \cos 2a} \end{array}\right\}$ if a be less than 45°.

18. $\dfrac{\mathrm{Sin}\, 2a + 2\sin^2 a - 2\sin a}{1 - \cos a} = 4\cos \tfrac{1}{2}a (\cos \tfrac{1}{2}a - \sin \tfrac{1}{2}a)$.

19. $\dfrac{\mathrm{Tan}\, a + \sec a - 1}{1 + \tan a - \sec a} = \tan a + \sec a$, and $\dfrac{\tan a + \sin a}{\mathrm{vers}\, a + \mathrm{chd}\, a}$
$= 2\dfrac{\cos \frac{1}{2}a}{\cos a} - \tan a$: also, $\dfrac{\tan a - \sin a}{\sin^3 a} = \dfrac{1}{\cos a (1 + \cos a)}$.

20. $\dfrac{\mathrm{Cos}\, a + \sin a - \frac{1}{2}\sin 2a - \sin^2 a}{2\cos 2a + 2\sin^2 a + \sin a - \sin^2 a} = \dfrac{\sin a + \cos a}{3\sin a + 2\cos^2 a}$.

21. $\mathrm{Sin}\, a + \sin 2a + \sin 3a = \sin 2a (3 - 4\sin^2 \tfrac{1}{2}a)$.

22. $\operatorname{Cos} \alpha + \cos 3\alpha + \cos 5\alpha = \cos 3\alpha\,(3 - 4 \sin^2 \alpha)$.

23. $1 + \cos \alpha + \cos 2\alpha + \cos 3\alpha = 2 \cos \alpha\,(2 \cos^2 \alpha + \cos \alpha - 1)$.

24. $\operatorname{Tan} \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha = \cot \alpha - 8 \cot 8\alpha$.

25. $(\operatorname{Cos} \alpha + \sqrt{-1} \sin \alpha - 1)(\cos \alpha - \sqrt{-1} \sin \alpha - 1)$
$= 4 \sin^2 \tfrac{1}{2}\alpha$.

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1. $\operatorname{Sin} \alpha = \cos(30^\circ - \alpha) - \cos(30^\circ + \alpha)$

$$= \frac{1}{\sqrt{3}}\left\{\sin(30^\circ + \alpha) - \sin(30^\circ - \alpha)\right\}$$

$$= \frac{1}{\sqrt{2}}\left\{\sin(45^\circ + \alpha) - \sin(45^\circ - \alpha)\right\} = 2 \sin^2(45^\circ + \tfrac{1}{2}\alpha) - 1$$

$$= \frac{1 - \tan^2(45^\circ - \frac{1}{2}\alpha)}{1 + \tan^2(45^\circ - \frac{1}{2}\alpha)} = \frac{\tan(45^\circ + \frac{1}{2}\alpha) - \tan(45^\circ - \frac{1}{2}\alpha)}{\tan(45^\circ + \frac{1}{2}\alpha) + \tan(45^\circ - \frac{1}{2}\alpha)}.$$

2. $\operatorname{Cos} \alpha = \sin(30^\circ + \alpha) + \sin(30^\circ - \alpha)$

$$= 2 \sin(45^\circ + \tfrac{1}{2}\alpha) \sin(45^\circ - \tfrac{1}{2}\alpha)$$

$$= \frac{1}{\sqrt{3}}\left\{\sin(60^\circ + \alpha) + \sin(60^\circ - \alpha)\right\} = \frac{2}{\tan(45^\circ + \frac{1}{2}\alpha) + \cot(45^\circ + \frac{1}{2}\alpha)}.$$

3. $\operatorname{Tan} \alpha = \tfrac{1}{2} \tan(45^\circ + \tfrac{1}{2}\alpha) - \tfrac{1}{2} \tan(45^\circ - \tfrac{1}{2}\alpha)$

$$= \frac{1 - \tan(45^\circ - \alpha)}{1 + \tan(45^\circ - \alpha)} = \frac{\tan(45^\circ + \frac{1}{2}\alpha) + \cot(45^\circ + \frac{1}{2}\alpha)}{\tan \frac{1}{2}\alpha + \cot \frac{1}{2}\alpha}$$

$$= \frac{\sqrt{\tan(45^\circ + \alpha)} - \sqrt{\tan(45^\circ - \alpha)}}{\sqrt{\tan(45^\circ + \alpha)} + \sqrt{\tan(45^\circ - \alpha)}}.$$

4. $4 \sin^2(30^\circ \pm \alpha) = 2 \sin^2 \alpha \pm \sqrt{3} \sin 2\alpha + 1$.

5. $2 \tan(30^\circ - \alpha) = \cot(30^\circ + \tfrac{1}{2}\alpha) - \tan(30^\circ + \tfrac{1}{2}\alpha)$.

6. $2 \cot(60^\circ - \alpha) = \tan(60^\circ + \tfrac{1}{2}\alpha) - \cot(60^\circ + \tfrac{1}{2}\alpha)$.

7. $\operatorname{Tan}(30^\circ + \tfrac{1}{2}\alpha) \tan(30^\circ - \tfrac{1}{2}\alpha) = \dfrac{2 \cos \alpha - 1}{2 \cos \alpha + 1}$.

8. $\operatorname{Tan}(45^\circ - \alpha) = \dfrac{\cos 2\alpha}{1 + \sin 2\alpha} = \sqrt{\dfrac{1 - \sin 2\alpha}{1 + \sin 2\alpha}}$.

9. $\operatorname{Sin} 3\alpha = 4 \sin \alpha \sin(60^\circ - \alpha) \sin(60^\circ + \alpha)$.

10. $\operatorname{Cos} 3\alpha = 4 \cos \alpha \sin(30^\circ - \alpha) \sin(30^\circ + \alpha)$.

11. $\dfrac{\operatorname{Sin} 3\alpha + \cos 3\alpha}{\sin 3\alpha - \cos 3\alpha} = \left(\dfrac{2 \sin 2\alpha + 1}{2 \sin 2\alpha - 1}\right) \tan(45^\circ - \alpha)$.

12. $3 \cot 3\alpha = \cot \alpha - \cot(60^\circ - \alpha) + \cot(60^\circ + \alpha)$.

13. $2 \text{ vers } (90^\circ + \frac{1}{2}\alpha) \text{ vers } (90^\circ - \frac{1}{2}\alpha) = \text{vers } (180^\circ - \alpha)$.

14. $\text{Sin } \beta = \sin (\alpha + \beta) \cos \alpha - \cos (\alpha + \beta) \sin \alpha$.

15. $\text{Cos } \alpha + \cos (\alpha + 2\beta) = 2 \cos (\alpha + \beta) (2 \cos^2 \frac{1}{2}\beta - 1)$.

16. $\text{Sin } (\alpha + \beta) \sin 3 (\alpha - \beta) = \sin^2 (2\alpha - \beta) - \sin^2 (2\beta - \alpha)$.

17. $\text{Sin}^2 \beta + \sin^2 (\alpha - \beta) + 2 \sin \beta \sin (\alpha - \beta) \cos \alpha = \sin^2 \alpha$.

18. $\text{Sec } \alpha \sec 2\alpha + \sec 2\alpha \sec 3\alpha = (\tan 3\alpha - \tan \alpha) \operatorname{cosec} \alpha$.

19. $\text{Tan } (\alpha + \beta) = \dfrac{\sin^2 \alpha - \sin^2 \beta}{\sin \alpha \cos \alpha - \sin \beta \cos \beta}$.

20. $1 + \cos 2\alpha \cos 2\beta = \cos^2 (\alpha + \beta) + \cos^2 (\alpha - \beta)$.

21. $\text{Cos}^2 (\alpha + \beta) - \sin^2 \alpha = \cos \beta \cos (2\alpha + \beta)$.

22. $\text{Sin}^2 (\alpha + \beta) = \sin^2 \alpha + \sin^2 \beta + 2 \sin \alpha \sin \beta \cos (\alpha + \beta)$.

23. $\text{Tan } \alpha - \tan \beta = \tan \alpha \tan (\alpha - \beta) (\cot \alpha + \tan \beta)$.

24. $\text{Tan}^2 \alpha - \tan^2 \beta = \sin (\alpha + \beta) \sin (\alpha - \beta) \sec^2 \alpha \sec^2 \beta$.

25. $\text{Sin } \alpha + \cos \beta = \{\sin \frac{1}{2}(\alpha + \beta) + \cos \frac{1}{2}(\alpha + \beta)\}$
$\times \{\sin \frac{1}{2}(\alpha - \beta) + \cos \frac{1}{2}(\alpha - \beta)\}$.

26. $1 - \cos 2\alpha \cos 2\beta = 2 (\sin^2 \alpha \cos^2 \beta + \cos^2 \alpha \sin^2 \beta)$.

27. $1 + \cos 2\alpha \cos 2\beta = 2 (\sin^2 \alpha \sin^2 \beta + \cos^2 \alpha \cos^2 \beta)$.

28. $\dfrac{\text{Cos } \beta - \sin \alpha}{\cos \beta + \sin \alpha} = \tan \{45^\circ - \frac{1}{2}(\alpha + \beta)\} \tan \{45^\circ - \frac{1}{2}(\alpha - \beta)\}$.

29. $\dfrac{1 + \operatorname{cosec}^2 \alpha \tan^2 \theta}{1 + \operatorname{cosec}^2 \beta \tan^2 \theta} = \dfrac{1 + \cot^2 \alpha \sin^2 \theta}{1 + \cot^2 \beta \sin^2 \theta}$.

30. $\text{Cos } (\alpha - \theta) \sin (\beta - \gamma) + \cos (\gamma - \theta) \sin (\alpha - \beta)$
$= \cos (\beta - \theta) \sin (\alpha - \gamma)$.

31. $\text{Sin } (\alpha + \beta) \sin (\beta + \gamma) = \sin \alpha \sin \gamma + \sin \beta \sin (\alpha + \beta + \gamma)$.

32. $\text{Sin } (\alpha + \beta) \sin (\beta + \gamma + \delta)$
$= \sin \alpha \sin (\gamma + \delta) + \sin \beta \sin (\alpha + \beta + \gamma + \delta)$.

33. $\text{Sin } (\alpha - \beta) \sin \gamma - \sin (\alpha - \gamma) \sin \beta = \sin (\gamma - \beta) \sin \alpha$.

34. $\text{Sin}^2 (\alpha - \beta) + \sin^2 (\alpha - \gamma) + \sin^2 (\beta - \gamma)$
$= 2 \{1 - \cos (\alpha - \beta) \cos (\alpha - \gamma) \cos (\beta - \gamma)\}$.

35. $\text{Sin } \alpha + \sin \beta + \sin \gamma$
$= 4 \sin \frac{1}{2} (\alpha + \beta) \sin \frac{1}{2} (\alpha + \gamma) \sin \frac{1}{2} (\beta + \gamma) + \sin (\alpha + \beta + \gamma)$.

36. $\text{Cos } \alpha + \cos \beta + \cos \gamma$
$= 4 \cos \frac{1}{2} (\alpha + \beta) \cos \frac{1}{2} (\alpha + \gamma) \cos \frac{1}{2} (\beta + \gamma) - \cos (\alpha + \beta + \gamma)$.

37. $2\sin(\alpha+\beta+\gamma)=\cos\alpha\sin(\beta+\gamma)+\cos\beta\sin(\alpha+\gamma)$
$+\cos\gamma\sin(\alpha+\beta)-2\sin\alpha\sin\beta\sin\gamma.$

38. $2\cos(\alpha+\beta+\gamma)=2\cos\alpha\cos\beta\cos\gamma-\sin\alpha\sin(\beta+\gamma)$
$-\sin\beta\sin(\alpha+\gamma)-\sin\gamma\sin(\alpha+\beta).$

39. $\text{Tan}\,\alpha+\tan\beta+\tan\gamma$
$$=\tan\alpha\tan\beta\tan\gamma+\frac{\sin(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma}.$$

40. $\text{Cot}\,\alpha+\cot\beta+\cot\gamma$
$$=\cot\alpha\cot\beta\cot\gamma-\frac{\cos(\alpha+\beta+\gamma)}{\sin\alpha\sin\beta\sin\gamma}.$$

41. $1+\sin\alpha\sin\beta+\sin\alpha\sin\gamma+\sin\beta\sin\gamma-\cos\alpha\cos\beta\cos\gamma$
$=2\{\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}\gamma+\cos\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}\gamma\}^2.$

42. $\text{Sin}(\alpha+\beta-2\gamma)\cos\beta-\sin(\alpha+\gamma-2\beta)\cos\gamma$
$=\sin(\beta-\gamma)\{\cos(\alpha+\beta-\gamma)+\cos(\alpha+\gamma-\beta)+\cos(\beta+\gamma-\alpha)\}.$

43. $(\text{Sin}\,\alpha\sin\beta-\sin\gamma\sin\delta)^2-(\sin\alpha+\sin\beta-\sin\gamma-\sin\delta)$
$\times\{\sin\alpha\sin\beta(\sin\gamma+\sin\delta)-\sin\gamma\sin\delta(\sin\alpha+\sin\beta)\}$
$=16\sin\tfrac{1}{2}(\alpha-\gamma)\cos\tfrac{1}{2}(\alpha+\gamma)\sin\tfrac{1}{2}(\alpha-\delta)\cos\tfrac{1}{2}(\alpha+\delta)$
$\times\sin\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\beta+\gamma)\sin\tfrac{1}{2}(\beta-\delta)\cos\tfrac{1}{2}(\beta+\delta).$

44. $\text{Cos}(\alpha+\beta+\gamma)+\cos(\alpha+\beta-\gamma)+\cos(\alpha-\beta+\gamma)$
$+\cos(\beta+\gamma-\alpha)=4\cos\alpha\cos\beta\cos\gamma.$

45. In the series, $\cot\alpha+\operatorname{cosec}\alpha+\tfrac{1}{2}\cot\tfrac{1}{2}\alpha+\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}\alpha$
$$+\frac{1}{2^2}\cot\frac{1}{4}\alpha+\frac{1}{2^2}\operatorname{cosec}\frac{1}{4}\alpha+\&\text{c.}:$$
shew that the terms, beginning with the third, are alternately an Arithmetic and Geometric mean, between the two preceding terms.

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1. If $\alpha+\beta+\gamma=180^\circ$, it is required to prove that

(1) $\text{Sin}\,\alpha-\sin\beta+\sin\gamma=4\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma.$

(2) $\text{Sin}\,2\alpha+\sin2\beta+\sin2\gamma=4\sin\alpha\sin\beta\sin\gamma.$

(3) $\text{Sin}^2\tfrac{1}{2}\alpha+\sin^2\tfrac{1}{2}\beta+\sin^2\tfrac{1}{2}\gamma+2\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma=1.$

(4) $\text{Sin}(\alpha+\beta)\sin(\beta+\gamma)=\sin\alpha\sin\gamma.$

(5) $\text{Cos}\,\alpha+\cos\beta+\cos\gamma=1+4\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma.$

(6) $\text{Cot}\,\alpha\cot\beta+\cot\alpha\cot\gamma+\cot\beta\cot\gamma=1.$

(7) $\text{Tan}\,\tfrac{1}{2}\alpha\tan\tfrac{1}{2}\beta+\tan\tfrac{1}{2}\alpha\tan\tfrac{1}{2}\gamma+\tan\tfrac{1}{2}\beta\tan\tfrac{1}{2}\gamma=1.$

(8) $\text{Cot}\,\tfrac{1}{2}\alpha\cot\tfrac{1}{2}\beta\cot\tfrac{1}{2}\gamma=\cot\tfrac{1}{2}\alpha+\cot\tfrac{1}{2}\beta+\cot\tfrac{1}{2}\gamma.$

2. If $\alpha+\beta+\gamma=90°$, it is required to prove that

(1) $\mathrm{Cos}\,2\alpha+\cos 2\beta+\cos 2\gamma=1+4\sin\alpha\sin\beta\sin\gamma.$

(2) $\mathrm{Tan}\,\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$
$+\sec\alpha\sec\beta\sec\gamma.$

(3) $\mathrm{Tan}\,\tfrac{1}{2}\alpha+\tan\tfrac{1}{2}\beta+\tan\tfrac{1}{2}\gamma-\tan\tfrac{1}{2}\alpha\tan\tfrac{1}{2}\beta\tan\tfrac{1}{2}\gamma$
$=1-\tan\tfrac{1}{2}\alpha\tan\tfrac{1}{2}\beta-\tan\tfrac{1}{2}\alpha\tan\tfrac{1}{2}\gamma-\tan\tfrac{1}{2}\beta\tan\tfrac{1}{2}\gamma.$

(4) $\dfrac{\mathrm{Cos}\,\alpha+\sin\gamma-\sin\beta}{\cos\beta+\sin\gamma-\sin\alpha}=\dfrac{1+\tan\frac{1}{2}\alpha}{1+\tan\frac{1}{2}\beta}.$

3. If α, β, γ be in arithmetical progression: prove that

(1) $\mathrm{Sin}\,\alpha-\sin\gamma=2\sin(\alpha-\beta)\cos\beta.$

(2) $\mathrm{Tan}(\alpha-\beta)=\dfrac{\cos\alpha-\cos\gamma}{\sin\alpha+\sin\gamma}=-\dfrac{\sin\alpha-\sin\gamma}{\cos\alpha+\cos\gamma}.$

(3) $\dfrac{\mathrm{Tan}\,\beta}{\tan(\beta-\gamma)}=\dfrac{\sin\alpha+\sin\gamma}{\sin\alpha-\sin\gamma}=\dfrac{\tan\frac{1}{2}(\alpha+\gamma)}{\tan\frac{1}{2}(\alpha-\gamma)}.$

(4) $\mathrm{Sin}\,\alpha\sin\gamma=\{\sin\beta+\sin(\alpha-\beta)\}\{\sin\beta-\sin(\alpha-\beta)\}.$

(5) $(\mathrm{Sin}\,\alpha-\sin\gamma)\sin\beta=(\cos\gamma-\cos\alpha)\cos\beta.$

(6) $\dfrac{1}{\tan\alpha+\tan\gamma}-\dfrac{1}{2\tan\beta}=\dfrac{1}{\cot\alpha+\cot\gamma}-\dfrac{1}{2\cot\beta}.$

4. If $\alpha+\beta+\gamma=180°$, and the sines and cosines of α, β, γ be in arithmetical progression: prove respectively that

(1) $\mathrm{Sin}\,\tfrac{1}{2}\alpha\sin\tfrac{1}{2}(\beta-\gamma)=\sin\tfrac{1}{2}\gamma\sin\tfrac{1}{2}(\alpha-\beta).$

(2) $2\cos\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma=\cos^2\tfrac{1}{2}\beta.$

5. If $\cos(\alpha-\gamma)\cos\beta=\cos(\alpha-\beta+\gamma)$: prove that the tangents of α, β, γ are in harmonical proportion.

6. If $\tan\theta=\dfrac{b\sin\alpha}{a+b\cos\alpha}$: then, $\tan(\alpha-\theta)=\dfrac{a\sin\alpha}{b+a\cos\alpha}.$

7. If $\tfrac{1}{2}\tan\theta=\sqrt{m(m+1)}$: then, $\tan\tfrac{1}{2}\theta=\sqrt{\dfrac{m}{m+1}}.$

8. If $\sin\phi=m\sin(2\theta+\phi)$: then, $\tan(\theta+\phi)=\dfrac{1+m}{1-m}\tan\theta.$

9. If $1=\left(\dfrac{\sin\alpha}{\sin\beta}\right)^2+(\cos\alpha\cos\gamma)^2$: then, $\sin\gamma=\dfrac{\tan\alpha}{\tan\beta}.$

10. If $\cos\alpha=\cos\beta\cos\gamma$: it is required to prove that
$\tan\tfrac{1}{2}(\alpha+\beta)\tan\tfrac{1}{2}(\alpha-\beta)=\tan^2\tfrac{1}{2}\gamma.$

11. If $\sin(\alpha+\beta)\cos\gamma = 2\cos(\beta-\gamma)\sin\alpha$: then will
$$\cot\alpha - \cot\beta = 2\tan\gamma.$$

12. If $\cos\beta(m+\cos\alpha) = 1+\cos\alpha$: then will
$$\tan^2\tfrac{1}{2}\alpha + \tan^2\tfrac{1}{2}\beta = m\{\tan^2\tfrac{1}{2}\alpha - \tan^2\tfrac{1}{2}\beta\}.$$

13. If $\tan\frac{1}{2}\beta = \dfrac{1-\tan^3\alpha}{1+\tan^3\alpha}$: it is required to prove that
$$2\cot 2\alpha = \sqrt[3]{\sec\beta+\tan\beta} - \sqrt[3]{\sec\beta-\tan\beta}.$$

14. If $\dfrac{\cos A - \sin B\sin C}{\cos B\cos C} = \dfrac{\cos D - \sin E\sin F}{\cos E\cos F}$:

then will $\dfrac{\cos B\cos C}{\cos E\cos F} = \dfrac{\text{vers}\,A - \text{vers}\,(B-C)}{\text{vers}\,D - \text{vers}\,(E-F)}$.

15. If $\cos\delta = \sin A\sin\alpha + \cos A\cos\alpha\cos x$, and $\sin\frac{1}{2}\delta = \cos\phi\cos\frac{1}{2}(A+\alpha)$: it is required to shew that
$$\sin\phi = \frac{\cos\frac{1}{2}x}{\cos\frac{1}{2}(A+\alpha)}\sqrt{\cos A\cos\alpha}.$$

16. Prove that $4\sin S\sin(S-A)\sin(S-B)\sin(S-C) = 1-\cos^2A-\cos^2B-\cos^2C+2\cos A\cos B\cos C$, where $2S = A+B+C$.

17. If $\cos\theta = \sqrt{\dfrac{2}{3}}$, and $\cos\phi = \dfrac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}}$: it is equired to shew that $\theta+\phi = 60°$.

18. If $\sin\theta = \dfrac{1}{\sqrt{3}}$, and $\cos\phi = \dfrac{1}{\sqrt{2}} + \dfrac{1}{2\sqrt{3}}$: it is equired to prove that $\theta-\phi = 30°$.

19. If $\tan\theta = \dfrac{1}{\sqrt{3}}$, and $\tan\phi = \dfrac{1}{\sqrt{15}}$: then will in $(\theta-\phi) = \sin 60°\cos 36°$.

20. If $\tan\theta = \dfrac{\sqrt{2}+1}{\sqrt{2}-1}$, and $\tan\phi = \dfrac{1}{\sqrt{2}}$: $\theta-\phi = 45°$.

21. If $\tan\theta = \frac{1}{3}$, and $\tan\phi = \frac{1}{7}$: then will $2\theta+\phi = 45°$.

22. If $\tan\theta = \cos\alpha\tan\phi$: then will
$$\tan(\phi-\theta) = \frac{\tan^2\frac{1}{2}\alpha\sin 2\phi}{1+\tan^2\frac{1}{2}\alpha\cos 2\phi}.$$

23. If $\theta = \alpha+\beta$: prove that $p^2+q^2+2pq\cos\theta$
$$= (p\cos\alpha + q\cos\beta)^2 + (p\sin\alpha - q\sin\beta)^2.$$

24. If $\tan\alpha\tan\beta = 3$, and $a\sin\beta = b\sin\alpha$, and we assume $\tan 2\phi = \dfrac{2ab}{3(a^2-b^2)}$; then will $\tan^2\alpha = \dfrac{3a}{b}\cot\phi$, and $\tan^2\beta = \dfrac{3b}{a}\tan\phi$.

25. If $\cot 2\alpha = -\tan\beta$: then will $\tan(\alpha-\beta) = \cot\alpha$.

26. If $\tan\frac{1}{2}\alpha = \tan^3\frac{1}{2}\beta$, and $\tan\beta = 2\tan\phi$: then will $2\phi = \alpha+\beta$.

27. If $a\cos(x+\alpha) + b\cos(x+\beta) = c\cos(x+\gamma)$: then will $\cos(\alpha-\beta) = \dfrac{c^2-a^2-b^2}{2ab}$, and $\tan\gamma = \dfrac{a\sin\alpha + b\sin\beta}{a\cos\alpha + b\cos\beta}$.

III. *Trigonometrical Equations and Elimination.*

1. If $\sin\theta\cos\theta = \dfrac{2}{5}$; then $\sin\theta = \pm\dfrac{2}{\sqrt{5}}$, or $= \pm\dfrac{1}{\sqrt{5}}$:

$$\text{and } \cos\theta = \pm\frac{1}{\sqrt{5}}, \text{ or } = \pm\frac{2}{\sqrt{5}}.$$

2. If $8\sin\theta = 3\cos^2\theta$; then $\sin\theta = \frac{1}{3}$ and $\cos\theta = \pm\frac{2}{3}\sqrt{2}$.

3. If $m\sin\theta + n\cos\theta = p$; then will

$$\sin\theta = \frac{mp \pm n\sqrt{m^2+n^2-p^2}}{m^2+n^2}, \quad \cos\theta = \frac{np \mp m\sqrt{m^2+n^2-p^2}}{m^2+n^2}.$$

4. If $6\tan\theta + 12\cot\theta = 5\sqrt{3}\sec\theta$; then will $\tan\theta = \pm\sqrt{3}$, and $\sec\theta = \pm 2$; or, $\theta = \pm 60°$, &c.

5. If $25\sin\theta(\sin\theta - \cos\theta) = 4$; then will

$$\sin\theta = \pm\frac{4}{5}, \text{ or } = \pm\frac{1}{10}\sqrt{2}; \ \cos\theta = \pm\frac{3}{5}, \text{ or } = \mp\frac{7}{10}\sqrt{2}.$$

6. If $\cos 3\theta = \sin\theta\sin 2\theta$: $\sin\theta = \pm\frac{1}{6}\sqrt{6}$, $\cos\theta = \pm\frac{1}{6}\sqrt{30}$.

7. If $\tan 2\theta = m$; $\sin^2\theta = \dfrac{\sqrt{m^2+1}\mp 1}{2\sqrt{m^2+1}}$, $\cos^2\theta = \dfrac{\sqrt{m^2+1}\pm 1}{2\sqrt{m^2+1}}$.

8. If $\tan\theta = \sec\alpha - \tan\alpha$; then $\theta = \frac{1}{2}(90° - \alpha)$.

9. If $\tan\theta = \operatorname{cosec} 2\theta$; shew that $\theta = 45°$, or $= 90°$.

10. If $\tan\frac{1}{2}\theta = \operatorname{cosec}\theta - \sin\theta$; then $\sin\frac{1}{2}\theta = \pm\dfrac{\sqrt{5}\mp 1}{2\sqrt{2}}$.

11. If $\sin(\theta - \alpha) = \cos(\theta + \alpha)$; prove that $\theta = 45^\circ$.

12. If $\sin\theta = \sin\alpha \sin(\beta + \theta)$; then $\tan\theta = \dfrac{\sin\alpha \sin\beta}{1 - \sin\alpha \cos\beta}$.

13. If $\tan\theta = \sin\alpha \tan(\beta + \theta)$; it is required to prove that $\sin(2\theta + \beta) = \cot^2(45^\circ - \frac{1}{2}\alpha)\sin\beta$.

14. If $\sin(\alpha + \theta)\sin\theta = \cos^2\frac{1}{2}\alpha$; then $\theta = 90^\circ - \frac{1}{2}\alpha$.

15. If $m\sin 2\theta = n\cos 2\theta + m$; it is required to prove that $\sin\theta = \pm\dfrac{1}{\sqrt{2}}$, or $= \pm\dfrac{m \pm n}{\sqrt{2(m^2 + n^2)}}$.

16. If $\{\sqrt{1 + \sin\theta} - 1\}\{\sqrt{1 - \sin\theta} + 1\} = \tan\frac{1}{2}\alpha \sin\theta$; prove that $\theta = 0^\circ$, or $= 2\alpha$, or $= 360^\circ + 2\alpha$, &c.

17. If $\cos 2\theta = 4\cos\alpha \cos\theta$; it is required to prove that $\cos\theta = \dfrac{1}{\sqrt{2}}\tan(\frac{1}{2}\beta \pm 45^\circ)$, where $\tan\beta = \sqrt{2}\cos\alpha$.

18. If $\cos n\theta + \cos(n-2)\theta = \cos\theta$; prove that $\theta = \dfrac{60^\circ}{n-1}$.

19. If $\tan(\alpha + \theta) = n\operatorname{cosec} 2\alpha - \cot 2\alpha$; it is required to prove that $\tan\theta = \dfrac{n-1}{n+1}\cot\alpha$.

20. If $\cos 3\theta + \cos 2\theta + \cos\theta = 0$; it is required to prove that $\theta = 45^\circ$, or $= 120^\circ$, or $= 135^\circ$, &c.

21. If $4\sin\theta \sin 3\theta = 1$; prove that $\theta = 18^\circ$, &c.

22. If $\operatorname{cosec}^2\frac{1}{2}\theta - \sec^2\frac{1}{2}\theta = 2\sqrt{3}\sec^2\theta$; prove that $\sin\theta = \pm(\sqrt{3} - 1)$.

23. If $2\cos\alpha \sin(\alpha - \theta) = \dfrac{\cos^2\theta}{1 + \sin\theta}$; then, $\theta = 2\alpha - 90^\circ$.

24. If $\cos(\alpha + \theta) = \cos\alpha \sin\theta + \sin\beta$; prove that

$$\cos(\theta + \phi) = \frac{\sin\beta}{\cos\alpha}\cos\phi, \text{ where } \tan\phi = \frac{\sin(45^\circ + \alpha)}{\cos 45^\circ \cos\alpha}.$$

25. If $\operatorname{vers}(2\alpha - \theta) - \operatorname{vers}(\alpha - \theta) = m\{\operatorname{vers}\theta - \operatorname{vers}(\alpha - \theta)\}$; then will $\tan\theta = \dfrac{\sin\frac{3}{2}\alpha - m\sin\frac{1}{2}\alpha}{\cos\frac{3}{2}\alpha - m\cos\frac{1}{2}\alpha}$.

26. If $\sin(\beta + x) + \sin(2\alpha - \beta + x)$
$= (\cos^2\beta - \cos^2\alpha)\sin(\alpha + 2x) - \sin\alpha(\sin^2\beta - \sin^2\alpha)$; then will $\sec x = \sin(\alpha + \beta)\tan(\alpha - \beta)$.

27 If $\tan\alpha\tan x = \tan^2(\alpha + x) - \tan^2(\alpha - x)$; then will $\cos x = \pm(1 + \cos\alpha)$.

28. If $\tan\theta + 2\cot 2\theta = \sin\theta\,(1 + \tan\frac{1}{2}\theta\tan\theta)$; then will $\theta = \pm 45^\circ$, or $= (4r \pm 1)\,45^0$.

29. If $\sin 3\theta - 2\sin 2\theta + \sin^2\theta + 4\sin^3\theta = 0$; then will $\cos\theta = \frac{1}{17}(12 \pm 2\sqrt{2})$.

30. If $\sin 2\theta\cos 2\theta + 3\sin\theta\cos 3\theta = 0$; then will $\theta = 0^\circ$, or $= 90^\circ$; or, $\cos\theta = \pm\frac{1}{4}\sqrt{11}$.

31. If $\sin 4\theta + 4\sin 3\theta\cos\theta = 0$; then will $\theta = 0^\circ$, or $= 90^\circ$; or, $\sin\theta = \pm\frac{1}{3}\sqrt{6}$.

32. If $\cos(\alpha - \beta)\sin(\gamma - \theta) = \cos(\alpha + \beta)\sin(\gamma + \theta)$; then will $\tan\theta = \tan\alpha\tan\beta\tan\gamma$.

33. If $\cos\theta = \cos^2\alpha - \sin^2\alpha\sqrt{1 - c^2\sin^2\theta}$; then will $\tan\frac{1}{2}\theta = \pm\tan\alpha\sqrt{1 - c^2\sin^2\alpha}$.

34. If $1 + \tan\alpha\cos\theta = \sin\theta$; $\theta = 90^\circ$, or $= 90^\circ \pm 2\alpha$.

35. If $\sin\theta\sqrt{1 + \tan^2\alpha\tan^2\beta} + \cos\theta\sqrt{1 - \tan^2\alpha\tan^2\beta} = \tan\alpha + \tan\beta$, and $\cos 2\phi = \tan^2\alpha\tan^2\beta$; then will

$$\sin(\phi + \theta) = \frac{\sin(\alpha + \beta)}{\sqrt{2}\cos\alpha\cos\beta}.$$

36. If $3 + \cos 4\theta = 4\cos 2\theta + 2n\cos^3\theta\sin\theta$; then will $\theta = 0^\circ$, or $\tan\theta = \frac{1}{2}\sqrt[3]{2n}$.

37. If $\sin(2\theta + \alpha) - \sin(2\theta - \alpha) = \sin(\theta + \alpha) - \sin\alpha - \sin(\theta - \alpha)$; then will $\sin\theta = \pm\frac{1}{4}\sqrt{10 \mp 2\sqrt{5}}$.

38. If $\tan\theta + 2\tan(\alpha - \theta) = \tan(\alpha + \beta - \theta)$; then will

$$\tan\theta = \frac{\sin\beta - \sin\alpha\cos(\alpha + \beta)}{\sin\alpha\sin(\alpha + \beta)}.$$

39. If $\tan(\alpha + \theta)\tan(\alpha - \theta) = \dfrac{1 - 2\cos 2\alpha}{1 + 2\cos 2\alpha}$; then will $\theta = 30^\circ$, or $= 150^\circ$: &c.

40. If $\tan\theta + \cot\theta = 4$; then $\theta = (n + \frac{1}{4} \pm \frac{1}{6})\,180^\circ$.

41. If $\tan^3\theta = \tan(\theta - \alpha)$; $\theta = \frac{1}{4}\{\alpha + \sin^{-1}(3\sin\alpha)\}$.

42. If $\{1 + a\cos(\theta + \alpha)\}\sin\beta = (1 + a\cos\alpha)\sin(\theta + \beta)$ prove that $\tan\frac{1}{2}\theta = \cot\beta + a\cos(\alpha - \beta)\operatorname{cosec}\beta$.

43. If $\tan^2\theta + \tan^2\phi = 1\frac{1}{3}$, and $\cos^2\theta + \cos^2\phi = 1\frac{1}{4}$; then will $\theta = 30^\circ$, or $= 45^\circ$: $\phi = 45^\circ$, or $= 30^\circ$: &c.

44. If $\cos(\theta-\phi)=\sin(\theta+\phi)$, and $\frac{\sin^2\theta-\sin^2\phi}{\sin(\theta-\phi)}$ $=\cot(\theta-\phi)$; then will $\theta=45°$, and $\phi=-45°$: &c.

45. If $\sin(\theta-\phi)=\cos(\theta+\phi)=\frac{1}{2}$; then will $\theta=45°$, nd $\phi=15°$: &c.

46. If $\cos\theta+\cos\phi=\frac{1}{2}(\sqrt{2}+\sqrt{3})$, and $\cos 3\theta+\cos 3\phi$ $=-\frac{1}{2}\sqrt{2}$; then will $\theta=30°$, and $\phi=45°$: &c.

47. If $\theta+\phi=a$, and $\sin\theta : \sin\phi :: m : 1$; then will

$$\tan\tfrac{1}{2}(\theta-\phi)=\frac{m-1}{m+1}\tan\tfrac{1}{2}a.$$

48. If $\theta+\phi=a$, and $\cos\theta : \cos\phi :: m : 1$; then will

$$\tan\tfrac{1}{2}(\phi-\theta)=\frac{m-1}{m+1}\cot\tfrac{1}{2}a.$$

49. If $\theta+\phi=a$, and $\text{vers}\,\theta : \text{vers}\,\phi :: m : 1$; then vill

$$\tan\tfrac{1}{4}(\theta-\phi)=\frac{\sqrt{m}-1}{\sqrt{m}+1}\tan\tfrac{1}{4}a.$$

50. If $\theta+\phi=a$, and $\text{chd}\,\theta : \text{chd}\,\phi :: m : 1$; then vill

$$\tan\tfrac{1}{4}(\theta-\phi)=\frac{m-1}{m+1}\tan\tfrac{1}{4}a.$$

51. If $\theta+\phi=a$, and $\tan\theta : \tan\phi :: m : 1$; then vill

$$\sin(\theta-\phi)=\frac{m-1}{m+1}\sin a.$$

52. If $15(1-\cos 2\theta\cos 2\phi)=17\sin 2\theta\sin 2\phi$, and in $(\theta-\phi)+\frac{1}{2}\sqrt{\sin(\theta+\phi)\sin(\theta-\phi)}=(\tan\theta-1)\cos\theta\cos\phi$; hen will $\tan\theta=5$, or $=1$, and $\tan\phi=3$, or $=\frac{3}{5}$.

53. If $\cot^{-1}x-\cot^{-1}(x+2)=15°$, then will the alues of x be $\sqrt{3}$, and $-(2+\sqrt{3})$.

54. If $\text{vers}^{-1}\frac{x}{a}-\text{vers}^{-1}\frac{bx}{a}=\text{vers}^{-1}(1-b)$; then will

$$\frac{x}{a}=1\pm\sqrt{\frac{2b}{1+b}}.$$

55. If $\sec^{-1}a+\sec^{-1}\frac{x}{a}=\sec^{-1}b+\sec^{-1}\frac{x}{b}$; $x=\pm ab$.

56. *If* $\tan 2a=\frac{2\tan a}{1-4\tan^2 a}$: then the radius $=\frac{1}{2}$.

57. If $\tan\theta + \tan\phi + \tan\psi = 1 + \dfrac{4}{\sqrt{3}}$,

$$\tan\theta\tan\phi + \tan\theta\tan\psi + \tan\phi\tan\psi = 1 + \frac{4}{\sqrt{3}},$$

$$\text{and } \tan\theta\tan\phi\tan\psi = 1:$$

then will $\theta = 30^\circ$, $\phi = 45^\circ$, and $\psi = 60^\circ$: &c.

58. In the equation $\sin\theta\sin(2\alpha + \theta) + n\cos^2\alpha = 0$, prove that θ is *real* only when $\cos\alpha < \dfrac{1}{\sqrt{n+1}}$.

59. If $\sin^3\theta = \sin(\alpha - \theta)\sin(\beta - \theta)\sin(\gamma - \theta)$, where α, β, γ, are the angles of a triangle: then will $\cot\theta = \cot\alpha + \cot\beta + \cot\gamma$; $\operatorname{cosec}^2\theta = \operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta + \operatorname{cosec}^2\gamma$.

60. If u represent *any* root of the equation $x^n - 1 = 0$: it is required to prove that *two* roots of the equation $x^{2n} - 2x^n \operatorname{cosec} 2\alpha + 1 = 0$, are represented by $u(\tan\alpha)^{\pm\frac{1}{n}}$.

IV. *Problems on Triangles.*

1. In a triangle ABC, right-angled at C:

$$\sin^2 \tfrac{1}{2}A = \frac{c-b}{2c}, \quad \cos^2 \tfrac{1}{2}A = \frac{c+b}{2c}, \quad \text{and } \tan^2 \tfrac{1}{2}A = \frac{c-b}{c+b}.$$

2. $\operatorname{Sin} 2A = \dfrac{2ab}{a^2+b^2}$, $\cos 2A = \dfrac{b^2-a^2}{b^2+a^2}$, $\tan 2A = \dfrac{2ab}{b^2-a^2}$.

3. $\operatorname{Sin}(45^\circ \pm A) = \dfrac{b \pm a}{c\sqrt{2}}$, and $\tan(45^\circ \pm A) = \dfrac{b \pm a}{b \mp a}$.

4. $\operatorname{Sin}(A - B) = \dfrac{a^2 - b^2}{c^2}$, and $\tan(A - B) = \dfrac{a^2 - b^2}{2ab}$.

5. The area $= \frac{1}{4}c^2 \sin 2A = \frac{1}{2}a^2 \tan B = \frac{1}{2}b^2 \tan A$.

6. If a', b', be the segments of the hypothenuse, made by a line bisecting the right angle: then will

$$a'b'(a+b)^2 = ab\,(a^2+b)^2.$$

7. The area $= s(s-c) = (s-a)(s-b) = \frac{1}{2}ab$:

8. Prove that $r = s - c = \dfrac{(s-a)(s-b)}{s} = \dfrac{ab}{a+b+c}$:

$$R = \frac{abc}{4s(s-c)} = \frac{abc}{4(s-a)(s-b)} = \tfrac{1}{2}c, \text{ and } 2(R + r) = a + b.$$

9. In an equilateral triangle whose side is a:
$\sin A = \frac{1}{2}\sqrt{3}$, $S = \frac{1}{4}a^2\sqrt{3}$, $r = \frac{1}{6}a\sqrt{3}$, and $R = \frac{1}{3}a\sqrt{3}$.

10. In an isosceles triangle, wherein $b = a$:

$$\cos A = \frac{c}{2a},\ \text{vers}\ C = \frac{c^2}{2a^2},\ S = \tfrac{1}{4}c\sqrt{4a^2 - c^2},$$

$$r = \tfrac{1}{2}c\sqrt{\frac{2a-c}{2a+c}},\ \text{and}\ R = \frac{a^2}{\sqrt{4a^2 - c^2}}.$$

11. If $2\cos B \sin C = \sin A$: then will the triangle e isosceles.

12. If $\tan A \sin^2 B = \tan B \sin^2 A$: then will the riangle be isosceles, or right-angled at C.

13. In any triangle, it is required to prove that

$$\frac{\text{ers}\ A}{\text{ers}\ B} = \frac{a(s-b)}{b(s-a)},\ \text{and}\ \frac{\text{vers}\ (A+B)}{\text{vers}\ C} = \frac{s(s-c)}{(s-a)(s-b)}.$$

14. $\text{Tan}\ \frac{1}{2}A \tan \frac{1}{2}B = \frac{s-c}{s}$: $\tan \frac{1}{2}A \cot \frac{1}{2}B = \frac{s-b}{s-a}$:
$\text{in}\ \frac{1}{2}A \cos \frac{1}{2}B = \frac{s-b}{c}\cos \frac{1}{2}C$, and $\frac{\cot \frac{1}{2}A + \cot \frac{1}{2}C}{\cot \frac{1}{2}B + \cot \frac{1}{2}C} = \frac{b}{a}$.

15. $\frac{\text{Tan}\ \frac{1}{2}A - \tan \frac{1}{2}B}{\tan \frac{1}{2}A + \tan \frac{1}{2}B} = \frac{a-b}{c}$, and $\frac{1 + \tan \frac{1}{2}A \tan \frac{1}{2}B}{1 - \tan \frac{1}{2}A \tan \frac{1}{2}B}$
$= \frac{a+b}{c}$; also, $\cos \frac{1}{2}A \cos \frac{1}{2}B \sec \frac{1}{2}(A+B) = \frac{a+b+c}{2c}$.

16. $c^2 \sin (A - B) = (a^2 - b^2) \sin (A + B)$, and
$c^2 \cos (A - B) = 2ab + (a^2 + b^2) \cos (A + B)$.

17. $\text{Cos}\ A + \cos B = 2\frac{a+b}{c}\sin^2 \frac{1}{2}C$, and $\cot C - \cot A$
$= \frac{a^2 - c^2}{ac}\ \text{cosec}\ B$; also, $\cos B + \sin B \cot C = \frac{a}{c}$.

18. $$\text{Tan}\ \tfrac{1}{2}(A+B-C) = \frac{(a+c)\tan \frac{1}{2}B + (a-c)\cot \frac{1}{2}B}{2c}.$$

19. $\text{Tan}\ \frac{1}{2}(A + B - C) : \tan \frac{1}{2}(A - B + C) = b \cos C : c \cos B$
and $(c - b)\tan (\frac{1}{2}A + B) = (c + b) \tan \frac{1}{2}A$.

20. Prove that $c^2 = (a + b)^2 \sin^2 \frac{1}{2}C + (a - b)^2 \cos^2 \frac{1}{2}C$

$$= \frac{\{(a+b)\sin \frac{1}{2}C\}^2 - \{(a-b)\cos \frac{1}{2}C\}^2}{\cos (A - B)}.$$

21. The perpendicular, from the angle C upon the opposite side c, will be

$$=\frac{ab}{c}\sin C=\frac{a+b+c}{\cot\frac{1}{2}A+\cot\frac{1}{2}B}=\frac{a^2\sin B+b^2\sin A}{a+b}$$

$$=\frac{c\cos(A-B)-c\cos(A+B)}{2\sin C}=\left(\frac{2S}{\cot A+\cot B}\right)^{\frac{1}{2}}.$$

22. The distance of the foot of the same perpendicular from the middle of the base, will be

$$=\frac{c\sin(A-B)}{2\sin C}=\frac{c}{2}\left(\frac{\tan A-\tan B}{\tan A+\tan B}\right)=\frac{(a^2-b^2)\sin(A+B)}{2c\sin C}.$$

23. The area S of the triangle will be expressed by

$$\tfrac{1}{2}ab\sin C=\sqrt{s(s-a)\,bc}\,\sin\tfrac{1}{2}A=\sqrt{(s-b)(s-c)bc}\,\cos\tfrac{1}{2}A$$

$$=s(s-a)\tan\tfrac{1}{2}A=\tfrac{1}{2}a^2\,\frac{\sin B\sin C}{\sin(B+C)}=\tfrac{1}{2}a^2\,\frac{\sin B\sin C}{\sin A}$$

$$=\tfrac{1}{2}\sqrt[3]{a^2b^2c^2\sin A\sin B\sin C}=\frac{2abc}{a+b+c}\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C$$

$$=\tfrac{1}{2}(a^2-b^2)\,\frac{\sin A\sin B}{\sin(A-B)}=\frac{abc^2}{2(a^2-b^2)}\sin(A-B)$$

$$=\tfrac{1}{2}(a+b+c)^2\,\frac{\sin A\sin B\sin C}{(\sin A+\sin B+\sin C)^2}=\frac{a^2b^2(\cot A+\cot B)}{2c^2\operatorname{cosec}^2 C}$$

$$=\frac{(a+b+c)^2(\cos A+\cos B+\cos C-1)}{4(\sin A+\sin B+\sin C)}=\frac{a^2+b^2-c^2}{4\tan\frac{1}{2}(A+B-C)}$$

$$=\frac{1}{4}\sqrt{\frac{4(a^4+b^4+c^4)-(a^2+b^2+c^2)^2}{\cot^2 A+\cot^2 B+\cot^2 C}}=\frac{a^2+b^2+c^2}{4(\cot A+\cot B+\cot C)}$$

$$=\frac{(a+b+c)^2}{4(\cot\frac{1}{2}A+\cot\frac{1}{2}B+\cot\frac{1}{2}C)}=\frac{ab+ac+bc}{2(\operatorname{cosec}A+\operatorname{cosec}B+\operatorname{cosec}C)}.$$

24. The radius of the inscribed circle is expressed by

$$r=s\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B\tan\tfrac{1}{2}C=\frac{c\sin A\sin B}{\sin A+\sin B+\sin C}$$

$$=\frac{abc(\sin A+\sin B+\sin C)}{(a+b+c)^2}=\frac{\sqrt[3]{a^2b^2c^2\sin A\sin B\sin C}}{a+b+c}.$$

25. The radius of the circumscribed circle is

$$R=\frac{a+b+c}{2(\sin A+\sin B+\sin C)}=\sqrt[3]{\frac{abc}{8(\sin A\sin B\sin C)}}$$

$$=\frac{ab\sin C}{c\{\cos(A-B)-\cos(A+B)\}}=\frac{abc}{(a^2+b^2-c^2)}\tan\tfrac{1}{2}(A+B-C).$$

26. $\dfrac{R}{r}=\dfrac{abc}{4(s-a)(s-b)(s-c)}=\dfrac{\sin A+\sin B+\sin C}{2\sin A\sin B\sin C}$.

27. If any two sides of a triangle be bisected by lines from the opposite angles: prove that the distance of their intersection from $A=\frac{1}{3}\sqrt{a^2+4bc\cos A}$.

28. If l be the length of the line bisecting the angle C, and meeting the opposite side in E: then will

$$\tan AEC=\frac{a+b}{a-b}\tan\tfrac{1}{2}C,\quad \cos ACE=\frac{(a+b)l}{2ab},$$

$$S=\tfrac{1}{2}(l^2-\tfrac{1}{4}c^2)\tan C,\quad \cos B-\cos A=2\cos\tfrac{1}{2}C\cos AEC.$$

29. If straight lines be drawn bisecting the angles: and if α, β, γ be the distances of their intersection from A, B, C respectively: then will

$$\alpha^2\left(\frac{1}{c}-\frac{1}{b}\right)+\beta^2\left(\frac{1}{a}-\frac{1}{c}\right)+\gamma^2\left(\frac{1}{b}-\frac{1}{a}\right)=0.$$

30. If P, p be the perpendiculars from the extremities of the base of a triangle upon the line bisecting the vertical angle, at distances D, d from it: then will

$$Pp=(s-a)(s-b):\ Dd=s(s-c):\ S=Pd=pD.$$

31. The square described upon any side of a triangle, is equal to the sum of the rectangles contained by the two others, and their segments respectively cut off by perpendiculars let fall upon them, produced if necessary, from the opposite angles.

32. If α, β, γ be the distances of the angles A, B, C of a triangle from the point in which the lines bisecting the angles intersect, then will

$$\frac{1}{\alpha}\cos\tfrac{1}{2}A+\frac{1}{\beta}\cos\tfrac{1}{2}B+\frac{1}{\gamma}\cos\tfrac{1}{2}C=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

33. If H be the point referred to in the last problem, and CH produced meet AB in E: then

$$CH : CE :: a+b : a+b+c.$$

34. If o be the centre of the circle inscribed in the triangle ABC, and Co be produced to meet AB in D: then $Co : oD :: a+b : c$.

35. The rectangles of the segments of the perpendiculars, from the angles of a triangle upon the opposite sides, are equal for all the angles.

36. The area of any triangle is to the area of the triangle formed by joining the points where the lines bisecting the angles meet the opposite sides, as

$$(a+b)(a+c)(b+c) \;:\; 2abc.$$

37. The area of any triangle is to the area of the triangle formed by joining the points where the perpendiculars from the angles meet the opposite sides, as

$$1 \text{ to } 2 \cos A \cos B \cos C.$$

38. If perpendiculars from the angles of a triangle be let fall upon the opposite sides, the products of the *alternate* segments of the sides thus made will be

$$abc \cos A \cos B \cos C.$$

39. The products of the *alternate* segments of the sides of a triangle made by lines bisecting the opposite angles $= a^2b^2c^2 \div (a+b)(a+c)(b+c)$.

40. Three indefinite straight lines intersect in A, B, C: and any other straight line cuts AB in C', BC in A', and AC in B': prove that

$$AB' \,.\, BC' \,.\, CA' = A'B \,.\, B'C \,.\, C'A:$$

and that the product of the areas of the triangles

$$A'BC', \; B'CA', \; C'AB'$$

$$= \frac{(A'B' \,.\, B'C' \,.\, C'A' \sin A' \sin B' \sin C')^2}{8 \sin A \sin B \sin C}.$$

41. Three circles whose radii are r_1, r_2, r_3 respectively, touch each other externally, prove that the area of the triangle formed by joining their centres is

$$= \sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)}.$$

42. In any triangle $a = r(\cot \tfrac{1}{2} B + \cot \tfrac{1}{2} C) = 2 R \sin A$:

$$S = r^2 \cot \tfrac{1}{2} A \cot \tfrac{1}{2} B \cot \tfrac{1}{2} C = 2R^2 \sin A \sin B \sin C.$$

43. If AD, BE and CF be perpendiculars upon the sides of a triangle ABC, and D, E, F be joined, the triangles DEC, DBF and AFE are similar to ABC: and DEF is the triangle of least perimeter that can be inscribed in ABC.

44. In the triangle DEF just mentioned, prove that

$$2s = a\cos A + b\cos B + c\cos C:$$
$$S = bc\sin A\cos A\cos B\cos C:$$
$$r = b\cos A\cot B\cos C:$$
$$R = \frac{ab\sin C}{4c\sin A\sin B}.$$

45. If A', B', C' be the angles which the sides of a triangle subtend at the centre of the inscribed circle, then will $4\sin A'\sin B'\sin C' = \sin A + \sin B + \sin C$.

46. The radius of the circle described through any three of the centres of the four circles that can be drawn to touch the sides of a triangle, is equal to the diameter of the circumscribed circle.

47. If o be the centre of the circle inscribed in any triangle, and oA, oB be joined, then will

$$r' = \frac{c}{\cot\frac{1}{4}A + \cot\frac{1}{4}B}, \text{ and } R' = \frac{c}{2\cos\frac{1}{2}C}:$$

where r' and R' are the radii of the circles inscribed in and circumscribed about the triangle AoB.

48. If $b = a\sin C$, and $c = a\cos B$, prove that

$$A = 90^\circ, \text{ and } B = C = 45^\circ.$$

49. If the sides of a triangle be $a\cos A$, $b\cos B$, and $c\cos C$, then will its angles be the supplements of $2A$, $2B$ and $2C$, when $A + B + C = 180^\circ$.

50. In a triangle ABC, if $m = a + b$, shew that the two sides a and b are $= m\cos^2\frac{1}{2}\phi$ and $m\sin^2\frac{1}{2}\phi$,

$$\text{where } \sin\phi = \pm\frac{1}{m}\sqrt{m^2 - c^2}\sec\tfrac{1}{2}C.$$

51. If the middle points of the sides of a triangle be joined with the opposite angles, and R_1, R_2, &c. be the radii of the circles described about the six triangles so formed; r_1, r_2, &c. the radii of the circles inscribed in the same: then will

$$R_1R_3R_5 = R_2R_4R_6, \text{ and } \frac{1}{r_1} + \frac{1}{r_3} + \frac{1}{r_5} = \frac{1}{r_2} + \frac{1}{r_4} + \frac{1}{r_6}.$$

52. In any triangle, if $b - a = mc$, then will

$$\cos(A + \tfrac{1}{2}C) = m\cos\tfrac{1}{2}C,$$

$$\text{and } \cot\tfrac{1}{2}(B-A) = \frac{1 + m\cos B}{m\sin B}.$$

53. If d_1, d_2, d_3, d_4 be the distances of the centres of the four circles which touch the sides of a triangle, from that of the circumscribed circle, then

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 = 4R^2 + 2R(r_1 + r_2 + r_3 - r) = 12R^2.$$

54. If three circular arcs whose radii are 3, 2, 1, at their centres subtend angles of 60°, 90°, 120°, and intersect each other at their extremities: prove that the sides of the triangle formed by their chords are

$$3,\ 2\sqrt{2},\ \sqrt{3}: \text{ and its area} = \tfrac{1}{2}\sqrt{23}.$$

55. If the tangents of A, B, C, in a triangle be in arithmetical and geometrical proportion, prove that

$$\tan A \tan C = 3, \text{ and } \sin 2A = \text{the } \textit{ratio} \times \sin 2C.$$

56. If circles be inscribed in an isosceles triangle, each touching the one above, the one below, and the sides of the triangle, whose vertical angle is α; prove that their radii are in a geometrical progression whose ratio is $\tan^2(45° + \frac{1}{4}\alpha)$.

57. When three circles are inscribed in a triangle, so as to touch one another, prove that the radius of the circle contiguous to the angle A

$$= \frac{r}{2}\left\{\frac{(1 + \tan\frac{1}{4}B)(1 + \tan\frac{1}{4}C)}{1 + \tan\frac{1}{4}A}\right\},$$

where r is the radius of the inscribed circle.

58. If ρ_1, ρ_2, ρ_3 denote the radii of these three circles, it is required to prove that

$$\frac{\sqrt{\rho_1\rho_2} + \sqrt{\rho_1\rho_3} + \sqrt{\rho_2\rho_3}}{\sqrt{r_1r_2 + r_1r_3 + r_2r_3}} + \frac{\rho_1}{r_1} + \frac{\rho_2}{r_2} + \frac{\rho_3}{r_3} = 1,$$

where r_1, r_2, r_3 are the radii of the escribed circles.

59. The sides of a triangle are in arithmetical progression, and its area is to that of an equilateral triangle of the same perimeter as 3 : 5; prove that the sides are as 3, 5, 7, and the greatest angle = 120°.

60. In a triangle, the rectangle of the perpendiculars from the angles at the base to the line bisecting the vertical angle, is equal to the rectangle of the segments of the base, made by the contact of the inscribed circle.

61. In any right-angled triangle, the distance between the middle of the hypothenuse, and the point of contact of the inscribed circle, is equal to half the difference of the base and perpendicular.

62. If lines be drawn from the three angles of an equilateral triangle inscribed in a circle, to any point in its circumference, the longest of these lines is always equal to the sum of the two others.

63. In any plane triangle, the excess of the square of half the sum of the sides above the square of the line bisecting the base, is equal to the rectangle of the radii of the inscribed circle, and of the circle which touches the base and the sides produced.

64. If the sides of a triangle be bisected, and from the points of bisection, perpendiculars be drawn from the triangle to the circumscribed circle: the sum of the lines so drawn, with the radius of the inscribed circle, will be equal to the diameter of the circumscribed circle.

65. Prove that the sum of the squares of the ten lines joining the centres of the five circles, four of which touch the sides of the triangle, and the remaining one is the circumscribed circle, is equal to fifteen times the square of the diameter of the circumscribed circle.

66. The sum of the squares of the distances of the centre of the inscribed circle, from the angles of the triangle, is equal to $ab + ac + bc - \dfrac{6abc}{a + b + c}$.

67. If p_1, p_2, p_3 be the lengths of the perpendiculars drawn from a point within a triangle, so as to bisect the sides a, b, c respectively: then will

$$4\left\{\frac{a}{p_1} + \frac{b}{p_2} + \frac{c}{p_3}\right\} = \frac{abc}{p_1 p_2 p_3}.$$

68. If p_1, p_2, p_3 be the perpendiculars from any point within a triangle, upon the sides: P_1, P_2, P_3 the perpendiculars from the angular points upon the same sides respectively: then will

$$\frac{p_1}{P_1} + \frac{p_2}{P_2} + \frac{p_3}{P_3} = 1.$$

69. If three lines drawn from the angular points to a point within the triangle, *trisect* its area: prove that the square of the distance of that point from the angle $A = \frac{1}{9}(2b^2 + 2c^2 - a^2)$.

70. Through the point of intersection D, of two straight lines drawn from two angles of a triangle, to bisect the opposite sides, a straight line $EDFG$ is made to pass, meeting two sides in E and F, and the remaining side produced in G; then will $\frac{1}{DE} - \frac{1}{DF} = \frac{1}{DG}$.

71. If A, B, C be the angles of a triangle: a, b, c the sides respectively opposite to them: and α, β, γ the lengths of the perpendiculars upon a, b, c from the angles A, B, C; then will $a = \frac{\alpha \sin A + \beta \sin B + \gamma \sin C}{\sin B \sin C}$.

72. The angles of a plane triangle form a geometrical progression whose ratio is $\frac{1}{2}$: shew that the greatest side $= 2$ perimeter $\times \sin 12°51'25''\frac{5}{7}$.

73. The three sides of a triangle are in arithmetical progression, and one of the angles is a right angle; prove that the sides are as the numbers 3, 4, 5.

74. The triangle ABC has its angles A, B, C in the proportion of the numbers 2, 3, 4 respectively: prove that $\cos \frac{1}{2} A = \frac{a + c}{2b}$, and $b^2 = \frac{a(a + c)^2}{2a + c}$.

75. If from the angle A of a triangle ABC, there be drawn a perpendicular AD upon the side BC: and if from D there be drawn two perpendiculars DE, DF upon AB, AC: then will $AE \, . \, EB \cos^2 C = AF \, . \, FC \cos^2 B$.

76. If a_1, a_2 be the sides of a right-angled triangle, and d_1, d_2 the diameters of the circles inscribed in the triangles formed by joining the right angle and the middle point of the hypothenuse: then will

$$\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{a_1} + \frac{1}{a_2} + 2\sqrt{\frac{1}{a_1^2} + \frac{1}{a_2^2}}, \text{ and } \frac{1}{d_1} - \frac{1}{d_2} = \frac{1}{a_2} - \frac{1}{a_1}.$$

77. If the centre of the inscribed circle of a triangle be *fixed*, and α, β, γ represent the distances of its angles from any other fixed point; then will $a\alpha^2 + b\beta^2 + c\gamma^2$ be invariable for *all* situations of the triangle.

78. If a, b, c be the middle points of the respective ides of the triangle ABC: S, s the sums of the squares f the three triangles whose bases are the sides of the riangles ABC, abc, respectively, and common vertex ny point in the plane of the triangle ABC: then will

$$4S - 16s = (\text{area of the triangle } ABC)^2.$$

79. If a circle be described, touching the base of a riangle and the sides produced, and a second circle be nscribed in the triangle: prove that the points where he circles touch the base, are equidistant from its extremities, and that the distance between the points, where hey touch either one of the sides, is equal to the base.

80. If through any point O within a triangle, straight ines be drawn from the angles A, B, C to meet the opposite sides in a, b, c respectively; then $\frac{Oa}{Aa} + \frac{Ob}{Bb} + \frac{Oc}{Cc} = 1$.

81. In any triangle, the differences of the segments f a side, made by a perpendicular from the opposite ngle, by the contact of the inscribed circle, and by the ine bisecting the opposite angle, are in geometrical rogression.

82. If O be any point in a triangle ABC; D, E, F ny points in the sides BC, AC, AB respectively: shew hat if OD, OE, OF be joined, and Aa, Bb, Cc be drawn arallel to them, $\frac{OD}{Aa} + \frac{OE}{Bb} + \frac{OF}{Cc} = 1$.

83. Given the perimeter and angles of a triangle: rove that $a = \frac{s \sin \frac{1}{2} A}{\cos \frac{1}{2} B \cos \frac{1}{2} C}$: $s - a = s \tan \frac{1}{2} B \tan \frac{1}{2} C$: $s - b = s \tan \frac{1}{2} A \tan \frac{1}{2} C$, and $s - c = s \tan \frac{1}{2} A \tan \frac{1}{2} B$.

84. In a plane triangle, O is the centre of the circumscribed circle, P the intersection of the perpendiculars from the angles upon the opposite sides: then, OP livides into parts in the ratio of 2 : 1, *any* line joining an ngle with the middle of its opposite side.

85. Through any point within a triangle, let right ines be drawn parallel to the three sides: then they vill bound three parallelograms P_1, P_2, P_3, and three riangles T_1, T_2, T_3, respectively opposite to them: prove hat $P_1^2 = 4T_2T_3$, $P_2^2 = 4T_1T_3$, and $P_3^2 = 4T_1T_2$.

86. If in the three edges of a rectangular parallelopiped which meet in any angle, the distances of three points A, B, C from that angle be a, b, c: then will the area of the triangle ABC be $\frac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}$.

87. Three circles whose radii are a, b, c, touch each other externally: prove that the tangents at the points of contact meet in a point, whose distance from any one of them is $\left(\dfrac{abc}{a+b+c}\right)^{\frac{1}{2}}$.

88. Retaining the notation previously used, it is required to prove that,

$$(1)\qquad r_1r_2 + r_1r_3 + r_2r_3 = s^2 = \frac{r_1r_2r_3}{r}.$$

$$(2)\qquad Ac \,.\, cB + Ba \,.\, aC + Cb \,.\, bA = r(r_1 + r_2 + r_3).$$

$$(3)\qquad \frac{1}{Ac \,.\, cB} + \frac{1}{Ba \,.\, aC} + \frac{1}{Cb \,.\, bA} = \frac{1}{r^2}.$$

$$(4)\qquad 4R = r_1 + r_2 + r_3 - r = \frac{(r_1 + r_2)(r_1 + r_3)(r_2 + r_3)}{r_1r_2 + r_1r_3 + r_2r_3}.$$

89. If p_1, p_2, p_3 be the perpendiculars from the angles upon the opposite sides; then,

$$(1)\qquad \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

$$(2)\qquad R = \frac{2rr_1r_2r_3}{p_1p_2p_3} = \frac{2r_1r_2r_3}{p_1p_2 + p_1p_3 + p_2p_3}.$$

$$(3)\qquad p_1 = \frac{2r_2r_3}{r_2 + r_3}. \qquad (4)\qquad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{r_3}.$$

90. From the middle of the hypothenuse of a right-angled triangle, draw a line so as to be equally inclined to it and the base, and terminated in the base produced: and from the middle of this line draw a second, equally inclined to it and the base produced, and terminated in the base produced; and so on: prove that the length of the n^{th} line so drawn is equal to $\dfrac{a}{2^n}\operatorname{cosec}\dfrac{\alpha}{2^n}$, and that the last line continually approximates to $\dfrac{a}{\alpha}$, where a is the altitude of the triangle, and α the opposite angle.

V. *Problems on Quadrilaterals and Polygons.*

1. If a, b be two adjacent sides of a parallelogram, and α their included angle; then will the diagonals be expressed by $(a^2 + b^2 \pm 2ab \cos \alpha)^{\frac{1}{2}}$.

2. The side of an equilateral triangle inscribed in a circle : the side of a square inscribed in the same circle :: $\sqrt{3} : \sqrt{2}$; and the area of the triangle : area of the square :: $(\sqrt{3})^3 : (2\sqrt{2})^2$.

3. In a quadrilateral $ABCD$, whose angles A, C are right angles: prove that the area

$$=(s-a)(s-d)=(s-b)(s-c).$$

4. The square of the side of a regular pentagon inscribed in a circle, is equal to the sum of the squares of the sides of a regular hexagon and decagon, inscribed in the same circle.

5. Compare the areas of two regular decagons, one inscribed in, and the other circumscribed about, a circle: and shew that they are in harmonic proportion with the area of a regular circumscribed pentagon.

6. If from one of the angles of a rectangle, a perpendicular be let fall on the diagonal d, and from the point of intersection, perpendiculars p, q be let fall on the sides containing the opposite angle; then $p^{\frac{2}{3}} + q^{\frac{2}{3}} = d^{\frac{2}{3}}$.

7. The difference between the side and diagonal of a regular pentagon, is equal to the side of a regular pentagon, whose diagonal is equal to the side of the proposed pentagon.

8. If the diagonals of a quadrilateral intersect each other at an angle of 30°: prove that the area of the rectangle contained by the diagonals is equal to four times the area of the quadrilateral.

9. If r and R be the radii of the circles, inscribed in, and circumscribed about, a regular polygon of n sides, and a be the side of the polygon: then will

$$R + r = \tfrac{1}{2} a \cot \frac{\pi}{2n}.$$

10. In a regular pentagon, if chords be drawn subtending each of the angles: these by their intersection

will form an interior regular pentagon, whose side is equal to $\frac{1}{2}(3-\sqrt{5})\times$ the side of the proposed figure.

11. The area of a regular polygon of n sides inscribed in a circle : the area of another regular polygon of $3n$ sides inscribed in the same circle :: 1 : k; prove that $\cos\theta=\frac{\sqrt{k+3}}{2\sqrt{k}}$, if θ be the angle which a side of the latter polygon subtends at the centre of the circle.

12. In any quadrilateral figure, the sum of the squares of the diagonals, together with four times the square of the line joining their middle points, is equal to the sum of the squares of all the sides.

13. If a point be assumed within a regular polygon of n sides, from which perpendiculars are drawn to each of the sides, produced if necessary: the sum of these perpendiculars : to the radius of the inscribed circle :: n : 1.

14. If the diagonals of a supplemental quadrilateral, intersect at right angles: their segments are proportional to the rectangles of the sides which are terminated at their extremities.

15. One regular figure has twice as many sides as another, and each of its angles greater than each of the angles of the other, in the ratio of 4 : 3; find the number of sides of each. Answer: 10 and 5 sides.

16. The number of sides of one regular polygon exceeds the number of those of another by 1, and an angle of one exceeds an angle of the other by 4°: find the number of sides of each. Answer: 10 and 9 sides.

17. The interior angles of a rectilineal figure are in arithmetical progression, the least angle being 120°, and the common difference 5°: find the number of sides, and explain the double answer.

Answer: 9 sides, and the incongruous value is 16.

18. One regular polygon has two sides more than another: and their angles are in the ratio of 9 : 8; what are the polygons? Answer: Octagons and Hexagons.

19. The numbers of sides of two regular polygons are as 3 : 4; and the difference of an angle of each is 10°: what are the polygons?

Answer: Nonagons and Dodecagons.

20. The greater the number of sides of a regular polygon, the greater is the magnitude of each of its angles: and the angle of the polygon approximates to two right angles as its limit, when the number of sides is indefinitely increased.

21. Prove that, in a regular dodecagon, the distance of two opposite sides, is equal to the side of the figure, together with that of its inscribed square.

22. If squares be described upon the sides of any triangle, and the angular points of the squares be joined: the sum of the squares of the sides of the hexagonal figure thus formed, is equal to four times the sum of the squares of the sides of the triangle.

23. If all the angular points of a regular polygon be joined, and r be the radius of the circumscribed circle: prove that the sum of all the lines including the perimeter of the polygon, is equal to $nr \cot \dfrac{90^\circ}{n}$.

24. In any right-angled triangle, the difference between the diameter of the inscribed circle and the side of the greatest inscribed square : the radius of the inscribed circle :: the diameter of that circle : the sum of the base and perpendicular.

25. If from any point in the circumference of a circle, straight lines be drawn to the angular points of an inscribed regular polygon of n sides: then will the sum of the squares of these lines $= 2n \times (\text{radius})^2$.

26. The interior angles of a decagon are in arithmetical progression, and the least angle is two-thirds of the greatest: find the least angle and the common difference.

Answer: 112°, and $6^\circ\ 13'\ 20''$.

27. If s_1 be the sum of the tangents of the angles of a polygon, s_3 the sum of their products taken three together, &c.: then will $s_1 - s_3 + s_5 - s_7 + \&c. = 0$.

28. If p be the perpendicular from the centre of a circle whose radius is r, upon the side of the regular inscribed *pentagon, and d* be the semiside of the regular *inscribed decagon*, then $r^3 = 2(p^3 - d^3)$.

29. In any polygon with n sides, A_1A_2, A_2A_3, &c., represented respectively by a_1, a_2, &c.: prove that

$$a_1 \sin A - a_2 \sin (A_1 + A_2) + a_3 \sin (A_1 + A_2 + A_3) - \&c.$$
$$\pm a_{n-1} \sin (A_1 + A_2 + A_3 + \&c. + A_{n-1}) = 0.$$

30. If $ABCDEF$ be a regular hexagon, and AC, BD, CB, DF, EA, FB be joined, meeting in the points a, b, c, d, e, f; shew that $abcdef$ is a regular hexagon, whose area is equal to one third of the area of the original hexagon $ABCDEF$.

31. If a straight line bisect at right angles, any side AB of a regular polygon of $2n + 1$ sides: shew that it must pass through the point of intersection of the two sides of the polygon, which are most remote from AB: and prove that the length of the part of the bisecting line within the polygon is equal to $\frac{1}{2} AB \cot \dfrac{90^\circ}{2n+1}$.

32. If S be any point in the diameter AB of a circle, whose circumference is divided into $2n$ equal parts; and lines be drawn from S to all the points of section: then will the sum of their squares be equal to $n(SA^2 + SB^2)$.

33. If a_{n-1}, a_n and a_{n+1} be the areas of regular polygons of 2^{n-1}, 2^n and 2^{n+1} sides, inscribed in a circle whose radius is r: then will $2a_n a^2_{n+1} = 2^{2n-1} r^4 (a_n - a_{n-1})$.

34. If A_1, A_2, A_3, &c., A_{2n} be the angles of a polygon of $2n$ sides, inscribed in a circle: then will

$$A_1 + A_3 + \&c. + A_{2n-1} = A_2 + A_4 + \&c. + A_{2n}.$$

35. If in a regular polygon of n sides, straight lines be drawn from the extremities of a side a, to those of any other side, so as to cross each other: then will the distance of each point of intersection, from the centre of the polygon, be $a \cot \dfrac{360^\circ}{n}$.

36. If R, r be the radii of the circumscribed and inscribed circles of a regular polygon of n sides, and R', r' the corresponding radii for a regular polygon of $2n$ sides, and of the same perimeter as the former: then will $Rr' = R'^2$, and $R + r = 2r'$.

37. If p, P be the perimeters of the inscribed and circumscribed polygons of n sides: p', P' those of the

nscribed and circumscribed polygons of $2n$ sides, of any ircle : then will

$$p'^2 = pP', \; p' = p\sqrt{\frac{2P}{P+p}}, \text{ and } P' = \frac{2Pp}{P+p}.$$

38. In a regular polygon of n sides described about circle whose radius is r, if c be the distance of any point from the centre, and perpendiculars be drawn from his point to all the sides of the polygon: the area of he figure, formed by joining the feet of the adjacent perpendiculars, is expressed by $\frac{1}{2}n\{r^2 \sin \alpha + \frac{1}{2}c^2 \sin 2\alpha\}$, where α is the angle at the centre subtended by each side.

39. If two circles have the same centre, and a regular polygon be described about the outer circle: and from any point in the circumference of the inner, perpendiculars be drawn to all the sides of the polygon: he figure formed by joining the feet of every pair of contiguous perpendiculars will be of an invariable magnitude.

VI. *Miscellaneous Theorems and Problems.*

1. If $\alpha + \beta + \gamma + \&c. + \lambda = m\pi$, and the notation of Article (65) be retained: then will

$$s_1 + s_5 + \&c. = s_3 + s_7 + \&c.$$

2. If $\alpha + \beta + \gamma + \&c. + \lambda = (2m-1)\frac{\pi}{2}$: then will

$$1 + s_4 + \&c. = s_2 + s_6 + \&c.$$

3. If $\tan\theta = \dfrac{n \sin\phi}{1 + n\cos\phi}$: it is required to shew that

$$\theta = n\sin\phi - \tfrac{1}{2}n^2 \sin 2\phi + \tfrac{1}{3}n^3 \sin 3\phi - \&c.$$

4. From the equation, $\sin p = \sin P \sin(z+p)$: prove hat $p = \sin P \sin z + \frac{1}{2}\sin^2 P \sin 2z + \frac{1}{3}\sin^3 P \sin 3z + \&c.$

5. If $\tan\theta = n\tan\phi$; then we shall have

$$\theta = \phi + \left(\frac{n-1}{n+1}\right)\sin 2\phi + \tfrac{1}{2}\left(\frac{n-1}{n+1}\right)^2 \sin 4\phi + \&c.:$$

$$\phi = \theta - \left(\frac{n-1}{n+1}\right)\sin 2\theta + \tfrac{1}{2}\left(\frac{n-1}{n+1}\right)^2 \sin 4\theta - \&c.$$

6. If $\tan\theta = \cos\alpha \tan\phi$: then will

$$\theta = \phi - \tan^2 \tfrac{1}{2}\alpha \sin 2\phi + \tfrac{1}{2} \tan^4 \tfrac{1}{2}\alpha \sin 4\phi - \&c.$$

7. If $\sin\theta = \sin\alpha \sin(\beta + \theta)$: then will

$$\theta = \sin\alpha \sin\beta + \tfrac{1}{2}\sin^2\alpha \sin 2\beta + \tfrac{1}{3}\sin^3\alpha \sin 3\beta + \&c.$$

8. It is required to prove that

$$e^x + e^{-x} = 2\left(1 + \frac{2^2x^2}{\pi^2}\right)\left(1 + \frac{2^2x^2}{3^2\pi^2}\right) \&c.:$$

$$e^x - e^{-x} = 2x\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{2^2\pi^2}\right) \&c.$$

9. It is required to shew that

$$2\sin(\alpha \pm \beta\sqrt{-1}) = (e^\beta + e^{-\beta})\sin\alpha \pm (e^\beta - e^{-\beta})\sqrt{-1}\cos\alpha$$

$$2\cos(\alpha \pm \beta\sqrt{-1}) = (e^\beta + e^{-\beta})\cos\alpha \mp (e^\beta - e^{-\beta})\sqrt{-1}\sin\alpha.$$

10. Prove that $2\sin\tfrac{1}{2}\alpha \sin^2\dfrac{\alpha}{2^2} + 2^2 \sin\dfrac{\alpha}{2^2} \sin^2\dfrac{\alpha}{2^3} + \&c.$ to n terms $= \tfrac{1}{2}\left(2^n \sin\dfrac{\alpha}{2^n} - \sin\alpha\right)$.

11. Shew that the sum of n terms of the series,

$$\sec\alpha \sec 2\alpha + \sec 2\alpha \sec 3\alpha + \sec 3\alpha \sec 4\alpha + \&c.$$
$$= \operatorname{cosec}\alpha \{\tan(n+1)\alpha - \tan\alpha\}.$$

12. Prove that the sum of n terms of the series,

$$\sec\alpha \sec 3\alpha - \sec 2\alpha \sec 4\alpha + \sec 3\alpha \sec 5\alpha - \&c.$$
$$= \tfrac{1}{2}\sec\alpha \{\sec\alpha \sec 2\alpha \mp \sec(n+1)\alpha \sec(n+2)\alpha\}.$$

13. Shew that the sum of n terms of the series,

$$(2\sec 2\alpha)^2 + (2^2 \sec 2^2\alpha)^2 + (2^3 \sec 2^3\alpha)^2 + \&c.$$
$$= (2^{n+1} \operatorname{cosec} 2^{n+1}\alpha)^2 - (2\operatorname{cosec} 2\alpha)^2.$$

14. Prove that the sum of n terms of the series,

$$\operatorname{cosec}\alpha \operatorname{cosec} 2\alpha + \operatorname{cosec} 2\alpha \operatorname{cosec} 3\alpha + \operatorname{cosec} 3\alpha \operatorname{cosec} 4\alpha + \&c.$$
$$= \operatorname{cosec}\alpha \{\cot\alpha - \cot(n+1)\alpha\}.$$

15. The sum of n terms of the series,

$$\left(\frac{1}{2}\operatorname{cosec}\frac{\alpha}{2}\right)^2 + \left(\frac{1}{2^2}\operatorname{cosec}\frac{\alpha}{2^2}\right)^2 + \left(\frac{1}{2^3}\operatorname{cosec}\frac{\alpha}{2^3}\right)^2 + \&c.$$
$$= (\operatorname{cosec}\alpha)^2 - \left(\frac{1}{2^n}\operatorname{cosec}\frac{\alpha}{2^n}\right)^2.$$

16. The sum of n terms of the series,

$$\left(\frac{1}{2}\tan\frac{\alpha}{2}\right)^2+\left(\frac{1}{2^2}\tan\frac{\alpha}{2^2}\right)^2+\left(\frac{1}{2^3}\tan\frac{\alpha}{2^3}\right)^2+\&c.$$

$$=(\cot\alpha)^2-\left(\frac{1}{2^n}\cot\frac{\alpha}{2^n}\right)^2+2\left(1-\frac{1}{2^n}\right).$$

17. The sum of n terms of the series,

$$\left(\frac{1}{2}\tan\frac{\alpha}{2}\right)\left(\frac{1}{2}\sec\frac{\alpha}{2}\right)^2+\left(\frac{1}{2^2}\tan\frac{\alpha}{2^2}\right)\left(\frac{1}{2^2}\sec\frac{\alpha}{2^2}\right)^2+\&c.$$

$$=\cos\frac{\alpha}{2^n}\left(2^n\sin\frac{\alpha}{2^n}\right)^{-3}-\cos\alpha\ \mathrm{cosec}^3\alpha.$$

18. The sum of n terms of the series,

$$2\tan\frac{\alpha}{2}\left(\tan\frac{\alpha}{2^2}\right)^2+2^2\tan\frac{\alpha}{2^2}\left(\tan\frac{\alpha}{2^3}\right)^2+\&c.$$

$$=2\tan\frac{\alpha}{2}-2^{n+1}\tan\frac{\alpha}{2^{n+1}}.$$

19. The sum of the infinite series,

$$x\cos(\alpha+\beta)+\frac{x^2}{1.2}\cos(\alpha+2\beta)+\frac{x^3}{1.2.3}\cos(\alpha+3\beta)+\&c.$$

$$=e^{x\cos\beta}\cos(\alpha+x\sin\beta)-\cos\alpha.$$

20. The sum of the infinite series,

$$\sin(\alpha+\beta)+\frac{x^2}{1.2}\sin(\alpha+2\beta)+\frac{x^3}{1.2.3}\sin(\alpha+3\beta)+\&c.$$

$$=e^{x\cos\beta}\sin(\alpha+x\sin\beta)-\sin\alpha.$$

21. The sum of the infinite series,

$$x\cos(\alpha+\beta)+\frac{x(x-1)}{1.2}\cos(\alpha+2\beta)+\&c.$$

$$=(2\cos\tfrac{1}{2}\beta)^x\cos(\alpha+\tfrac{1}{2}x\beta)-\cos\alpha.$$

22. The sum of the infinite series,

$$x\sin(\alpha+\beta)+\frac{x(x-1)}{1.2}\sin(\alpha+2\beta)+\&c.$$

$$=(2\cos\tfrac{1}{2}\beta)^x\sin(\alpha+\tfrac{1}{2}x\beta)-\sin\alpha.$$

23. Prove *that the* roots of the equation $x^3-3x+1=0$, *are* $2\sin 10^\circ$, $2\sin 50^\circ$, and $-2\sin 70^\circ$.

24. Shew that the three roots of the equation,

$$x^3 - x^2 - 2x + 1 = 0,$$

are $2 \cos \frac{1}{7}\pi$, $-2 \cos \frac{2}{7}\pi$, and $2 \cos \frac{3}{7}\pi$.

25. Prove that the five roots of $x^5 - 1 = 0$, are 1,

$$\cos \tfrac{2}{5}\pi \pm \sqrt{-1} \sin \tfrac{2}{5}\pi, \quad \cos \tfrac{4}{5}\pi \pm \sqrt{-1} \sin \tfrac{4}{5}\pi.$$

26. Shew that the six roots of $x^6 - 1 = 0$, are ± 1,

$$\cos \tfrac{1}{3}\pi \pm \sqrt{-1} \sin \tfrac{1}{3}\pi, \quad \cos \tfrac{2}{3}\pi \pm \sqrt{-1} \sin \tfrac{2}{3}\pi.$$

27. AQB is a semicircle with its diameter horizontal, and ALP is a chord cutting the vertical radius CQ in L, so that $CL = \frac{1}{n} CQ$: shew that the area of the triangle $CLP = \frac{n^2 - 1}{n^2 + 1} \times$ the area of the triangle CLA.

28. ABC is a triangle inscribed in a circle, and a_1, a_2, a_3 are the angles which radii drawn from the centre to the angular points A, B, C make with the sides AB, BC, CA respectively, so related that $\tan a_1 = m \tan a_2$:

$\tan a_2 = m \tan a_3$: prove that $\tan a_3 = \dfrac{1}{\sqrt{m + m^2 + m^3}}$.

29. If i be the inclination of a plane to the horizon, and a the inclination of a line in it to the intersection of the plane with the horizontal plane: shew that the inclination θ of the line to the horizon, is found from the equation $\sin \theta = \sin a \sin i$.

30. If a, b, c be the lengths of the chords of three arcs of a circle which together make up a semicircumference, and r be the radius of the circle: then will

$$4r^3 - (a^2 + b^2 + c^2)\, r - abc = 0.$$

31. If the perimeter of a triangle be equal to four times the base c, and ρ be the distance of the centre of the inscribed circle from the angle A, making the angle ϕ with the base: then will $\rho\,(1 + \sin^2\phi) = c \cos \phi$.

32. Let one side AC of a triangle ABC be divided into any number of parts; and let lines be drawn through the points of division respectively parallel to AB, BC, and forming triangles whose areas are α, β, γ, &c.: then will the area of the $\triangle\, ABC = (\sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma} + \text{\&c.})^2$.

33. If in the arc AB of a circle, whose centre is O, the arc AC be made equal to the sine of AB, and the chord AB be drawn: then will the area of the sector BOC, be equal to the area of the segment ACB.

34. The figure ABC is formed of three *equal* arcs of the same circle, whose sum is equal to the semicircumference: prove that the area of this figure : the area of the circle :: 1 : 6.

35. The sides of a plane triangle are arcs of equal circles, radius $=1$: prove that the *algebraical* sum of its sides and angles is always equal to 180°, those sides being considered negative which are concave towards the interior of the triangle. If the three angles of such a triangle be given, investigate a formula for the determination of a side, and adapt it to logarithmic calculation.

36. Of all triangles of a given altitude and vertical angle, the base of the isosceles triangle is the least.

37. If α be the shortest line drawn from points in AC, AB so as to *bisect* the triangle ABC: and β, γ be corresponding lines drawn from the other sides; shew that $\alpha^2 \tan \frac{1}{2}B = \beta^2 \tan \frac{1}{2}A$: and that the area of the triangle $= \frac{1}{2}\sqrt{\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2}$.

38. If α_1, α_2, &c. be the arcs of a circle, whose radius $=1$, measured from the same point, each of which is equal to its tangent: prove that $\alpha_n + \dfrac{1}{\alpha_n} = (2n+1)\dfrac{\pi}{2}$, nearly.

39. If from the extremities of the base of a semicircle, two chords be drawn crossing each other, and making angles α, β with the base, and the extremities of the chords be joined: prove that the areas of the two triangles thus formed, will be as $1 : \cos^2(\alpha+\beta)$.

40. If through any point in the arc of a quadrant whose radius is R, two circles be drawn touching the bounding radii of the quadrant, and r, r' be the radii of these circles: shew that $rr' = R^2$.

41. A circle is inscribed in an equilateral triangle; an equilateral triangle in the circle: a circle again in the latter triangle, and so on: if r, r_1, r_2, r_3, &c. be the radii *of the circles: prove that* $r = r_1 + r_2 + r_3 +$ &c. in infinitum.

42. If a circle be inscribed in an equilateral triangle, and at the angular points, other circles be described, and so on: prove that the sum of the areas of *all* these circles $= \frac{11}{96}\pi a^2$, where a is a side of the triangle.

43. If a circle be inscribed in a given square, and between this circle and the angular points of the square, four other circles be inscribed, and the operation be repeated *in infinitum:* prove that the sum of the areas of all these circles $= \frac{2}{7}\pi a^2 (22\sqrt{2} - 31)$, where a is the side of the square.

44. If in either half of a rectangular parallelogram, a circle be described: prove that the straight lines drawn from its centre parallel to the sides, will cut off a gnomon equal to half the parallelogram.

45. If on the three sides of the triangle ABC, triangles similar and equal to the original triangle be drawn, in such a manner that the angles with their vertex at A shall be equal, and the same at B and C: and if in the three triangles so formed, circles be inscribed, and their centres be joined so as to form a triangle abc: also, if R, r be the radii of the circles circumscribed about ABC, abc, prove that $\dfrac{\text{area of } ABC}{\text{area of } abc} = \dfrac{R}{r}$.

46. If the sides of a triangle be three consecutive numbers, and the greatest angle be double of the least: shew that the magnitudes of the sides are as 4, 5, 6.

47. If p be the perpendicular upon the side c, from the opposite angle C: then will

$$2a = \sqrt{c^2 + 2pc \cot \tfrac{1}{2}C} \pm \sqrt{c^2 - 2pc \tan \tfrac{1}{2}C}:$$

$$2b = \sqrt{c^2 + 2pc \cot \tfrac{1}{2}C} \mp \sqrt{c^2 - 2pc \tan \tfrac{1}{2}C}.$$

48. A person, wishing to determine the length of an inaccessible wall, places himself due south of one end, and then due west of the other, at such distances that each of the angles which the wall subtends at the two positions $= \alpha^0$: prove that the length of the wall $= a \tan \alpha$, where a is the distance between the stations.

49. The shadows of two walls, which are at right angles to each other, and whose heights are a_1, a_2, are observed when the sun is due south, to be b_1 and b_2 in

breadth : if α be the sun's altitude, and β the inclination of the first wall to the meridian: prove that

$$\cot^2\alpha = \frac{b_1^2}{a_1^2} + \frac{b_2^2}{a_2^2}, \text{ and } \cot\beta = \frac{a_1 b_2}{a_2 b_1}.$$

50. If AB be produced to meet DCE in C: prove that the radii of the circles passing through A, B and touching DCE, will be $\dfrac{a + b \mp 2\sqrt{ab}\cos\alpha}{\sin 2\alpha}$, where $AC = a$, $BC = b$, and $\angle ACD = \alpha$.

51. If AB be the diameter of a circle, C the centre, FAG a tangent at A, ACF one third of a right angle, and FG equal to three times the radius: then BG being joined, will be very nearly equal to the semicircumference of the circle.

52. If one of the angles of a quadrilateral inscribed in a circle be a right angle, and if two opposite sides be produced to meet, the *Theorem* for the tangent of the sum of two angles, may be briefly deduced.

53. If two points, each 10 feet above the earth's surface, cease to be visible to each other at the distance of 8 miles, prove that the earth's radius = 4224 miles, nearly.

54. If the angle which the earth's radius r subtends at the sun be $8''.57116$, prove that the distance of the earth from the sun $= 24074r$, nearly.

55. Prove that the angle which a flag-staff 5 yards long, on the top of a tower 200 yards high, subtends at a point on the horizon 100 yards from the foot of the tower is $= 34'.22''.64$, nearly.

56. A flag-staff a feet long is placed on the top of a tower b feet high: prove that at the distance of $b\sqrt{\dfrac{a+b-2c}{a-b}}$ feet between the observer's eye whose height is c feet, and the foot of the tower, the staff and tower subtend equal angles.

57. A ship is approaching land, which is just visible from the mast-head, and after sailing x yards, land is just visible from the deck, prove that $x = (\sqrt{h} - \sqrt{h'})\sqrt{2r}$, *approximately;* when h and h', the heights of the mast-

head and deck, are very small compared to r, the radius of the earth.

58. A person in a balloon observed the angle between two places A and B bearing *north* and *south* of each other and a miles apart, to be α: and that from B *due east* of him to a point directly under him to be β; prove that his altitude was $a \cot \alpha \cos \beta$.

59. If A_1, a_1 be the areas or perimeters of similar polygons inscribed in and circumscribed about a circle; A_2, a_2 those of *twice* the number of sides: then $\frac{A_1 - a_1}{A_2 - a_2}$ is generally greater than 4, but *ultimately* equal to it.

60. If $m \sin \theta + n \cos \phi = p \cos \alpha$, $m' \cos \theta + n' \sin \phi = p' \sin \alpha$, prove that when $\theta \pm \phi = 2\alpha$, we shall have

$$m'p \cos \alpha \cos 2\alpha - mp' \sin \alpha \sin 2\alpha = np' \sin \alpha,$$

$$\text{and } m'p \cos \alpha \sin 2\alpha + mp' \sin \alpha \cos 2\alpha = \mp n'p \cos \alpha.$$

61. If $a = b + c$, it is required to prove that

$$\left\{\frac{1 - \cos ma}{ab} - \frac{1 - \cos mc}{bc}\right\}^2 + \left\{\frac{\sin ma}{ab} - \frac{\sin mc}{bc}\right\}^2$$
$$= \frac{1}{abc}\left\{\frac{a^2 + b^2 + c^2}{abc} + 2\frac{\cos ma}{a} - 2\frac{\cos mb}{b} - 2\frac{\cos mc}{c}\right\}.$$

62. By means of the solution of $x^{2n} - 1 = 0$, prove that the values of θ in the equation

$$\cos \theta + \cos 2\theta + \cos 3\theta + \&c. + \cos (n - 1)\theta = 0,$$

are expressed by $\frac{m\pi}{n}$: and by means of that of $x^{2n+1} \mp 1 = 0$, shew that the values of θ in the equation

$$\cos \theta + \cos 3\theta + \cos 5\theta + \&c. + \cos (2n - 1)\theta = 0,$$

are expressed by $\theta = \frac{2m\pi}{2n + 1}$, or $\theta = \frac{2m + 1}{2n + 1}\pi$.

63. By means of subsidiary angles, express $a + b + c + d + \&c.$, and $a - b + c - d + \&c.$ in the *forms*, $a \sec^2\theta \sec^2\theta_1 \sec^2\theta_2 \&c.$, and $a \cos^2\theta \sec^2\theta_1 \cos^2\theta_2 \&c.$, respectively.

64. Express $(a_1 + b_1)(a_2 + b_2)(a_3 + b_3) \&c.$ in the *form*, $a_1 a_2 a_3 \&c. \sec^2\theta_1 \sec^2\theta_2 \sec^2\theta_3 \&c.$: and $(a_1 - b_1)(a_2 - b_2)(a_3 - b_3) \&c.$ in the *form*, $a_1 a_2 a_3 \&c. \cos^2\theta_1 \cos^2\theta_2 \cos^2\theta_3 \&c.$

65. If s_n denote the sum of n terms of the series $a+b+c+d+\&c.$: prove that

$$\log s_n = \log s_2 + \log\left(1+\frac{c}{s_2}\right) + \log\left(1+\frac{d}{s_3}\right) + \&c.$$

66. If $x\cos\alpha + x'\sin\alpha = x\cos\beta + x'\sin\beta = y\cos\gamma + y'\sin\gamma = y\cos\delta + y'\sin\delta = 1$, the value of $(x+y)\div(x'+y')$ will be the *same*, however $\alpha, \beta, \gamma, \delta$ may be *interchanged*.

67. If a, b, c, d be the sides of a quadrilateral, whereof a is parallel to c: then will the distance between these parallels be

$$\frac{\sqrt{2(a-c)^2(b^2+d^2)-(a-c)^4-(b^2-d^2)^2}}{2(a-c)}.$$

68. If the angle A of a quadrilateral, whose sides are a, b, c, d, be a right angle: then will its area be

$$\tfrac{1}{2}ad + \sqrt{4(s-a)(s-b)(s-c)(s-d) - ad(ad+2bc)}.$$

69. In any quadrilateral figure $ABCD$, prove that

$$AB = \frac{CD\sin D + BC\sin(C+D)}{\sin A};$$

$$\text{and } \tan ABC = \frac{CD\sin C + AD\sin(C+D)}{BC + CD\cos C + AD\cos(C+D)}.$$

70. If upon one radius of a quadrant, a semicircle be described, and a circle be drawn to touch the arcs of the quadrant and semicircle and the other radius: twice its area will be equal to that of the semicircle.

71. If upon either radius of a quadrant, a semicircle be described, and another semicircle be drawn to touch this and the quadrantal arc at its extremity; shew that twice the area of the circle, in the curvilinear space thus formed, is equal to the area of the latter semicircle.

72. If two circles touch each other internally, and any circle be described touching both: prove that the sum of the distances of its centre from the centres of the two given circles, will be invariable. Also, if a series of such circles be described, touching each of the given circles and one another, and a, b, r be respectively the radii of the given circles and of the first circle in the series: prove that the radius of the $(n+1)^{\text{th}}$ circle will be

$$= \frac{ab(a-b)r}{abr + \{n(a-b)\sqrt{r} \pm \sqrt{ab(a-b-r)}\}^2}.$$

73. The radii of two circles, which intersect, are r, r', and c is the distance of their centres; prove that the length of their common chord

$$=\frac{1}{c}\left\{(r+r'+c)(r+r'-c)(r+c-r')(r'+c-r)\right\}^{\frac{1}{2}}.$$

74. The distance between the centres of two wheels $= a$, and the sum of their radii $= c$: prove that the length of a string, which crosses between the wheels, and is just wrapped round them both

$$=2\left\{\sqrt{a^2-c^2}+c\cos^{-1}\left(-\frac{c}{a}\right)\right\}.$$

75. A person standing at a known distance from two towers, in the same straight line with them, observes their apparent altitudes to be the same: but after walking a given distance towards them, he finds the angle of elevation of one of them to be double that of the other: find their heights.

76. $ABCD$ is a square, whose side CD is produced to any point E: a person holding EC horizontal, sees the top of a tree in the direction EB, and after moving through a given space, directly towards the tree, sees the top of it in the direction EA: required the height of the tree.

77. Find the sum of n terms of the series,

$$(\cos\alpha+\sqrt{-1}\sin\alpha)+(\cos\alpha+\sqrt{-1}\sin\alpha)^2+\&c.:$$

and thence deduce the sums of the series,

$$\sin\alpha+\sin 2\alpha+\sin 3\alpha+\&c. \text{ to } n \text{ terms}:$$

$$\cos\alpha+\cos 2\alpha+\cos 3\alpha+\&c. \text{ to } n \text{ terms}.$$

78. If $17\theta=\pi$, it is required to prove that

$$\cos\theta+\cos 3\theta+\cos 5\theta+\&c.+\cos 15\theta=\tfrac{1}{2}:$$

$$\cos\theta+\cos 9\theta+\cos 13\theta+\cos 15\theta=\tfrac{1}{4}(1+\sqrt{17}):$$

$$\cos 3\theta+\cos 5\theta+\cos 7\theta+\cos 11\theta=\tfrac{1}{4}(1-\sqrt{17}):$$

and if $s=\cos\theta+\cos 13\theta$, $t=\cos 9\theta+\cos 15\theta$,

$$u=\cos 3\theta+\cos 5\theta,\quad v=\cos 7\theta+\cos 11\theta:$$

then will $st=uv=-\frac{1}{4}$.

From these equations determine the values of s, t, u, v: and thence the side of a regular polygon of 17 sides, inscribed in a circle whose radius is r.

APPENDIX II.

ON SPHERICAL TRIGONOMETRY.

CHAPTER I.

MISCELLANEOUS THEOREMS AND PROBLEMS IN SPHERICAL TRIGONOMETRY, WITH THEIR SOLUTIONS.

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1. In the fundamental proposition of Article (25), the proof has been confined to the case where each of the sides is less than $\frac{1}{2}\pi$: it may easily be extended so as to include sides of all magnitudes, but the *Permanence of Equivalent Forms* renders it altogether unnecessary to go through every variety that may occur.

2. Since $\sin^2 A = 1 - \cos^2 A = 1 - \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)^2$

$= \frac{(1-\cos^2 b)(1-\cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$, we have

$$\frac{\sin^2 A}{\sin^2 a} = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c},$$

which, by reason of the *symmetry* of the latter member,

gives, $\frac{\sin^2 A}{\sin^2 a} = \frac{\sin^2 B}{\sin^2 b} = \frac{\sin^2 C}{\sin^2 c}$,

the result of Article (29), by a much shorter process.

The same relations are easily obtained from

$$\sin^2 a = 1 - \cos^2 a = 1 - \left(\frac{\cos A + \cos B \cos C}{\sin B \sin C}\right)^2.$$

3. From Article (25), we have immediately,

$$1 + \cos c = 1 + \cos a \cos b + \sin a \sin b \cos C$$

$$= \{1 + \cos(a-b)\}\cos^2 \tfrac{1}{2} C + \{1 + \cos(a+b)\}\sin^2 \tfrac{1}{2} C,$$

$$\therefore \cos^2 \tfrac{1}{2} C = \cos^2 \tfrac{1}{2}(a-b)\cos^2 \tfrac{1}{2} C + \cos^2 \tfrac{1}{2}(a+b)\sin^2 \tfrac{1}{2} C,$$

which *is analogous to* (3) of Article (111), Pl. Tr.

Also, $\sin^2 \frac{1}{2} c = \sin^2 \frac{1}{2}(a-b) \cos^2 \frac{1}{2} C + \sin^2 \frac{1}{2}(a+b) \sin^2 \frac{1}{2} C$; and these lead without much difficulty to the results of Article (43), by means of *Napier's* Analogies.

4. Tan $\frac{1}{2}(A+B+C) = \tan(\frac{1}{2}\Sigma + 90^\circ)$, by Article (74),

$$= -\cot \tfrac{1}{2} \Sigma = -\frac{1 + \cos a + \cos b + \cos c}{2\sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}}.$$

5. Sin $\frac{1}{2}(a+b+c) = \sin \frac{1}{2}\{3\pi - (A' + B' + C')\}$

$$= \frac{\sqrt{-\cos S \cos (S-A) \cos (S-B) \cos (S-C)}}{2 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C}.$$

6. Tan $\frac{1}{2}(A+B-C) = \dfrac{\tan \frac{1}{2} a \tan \frac{1}{2} b + \cos C}{\sin C}$

$$= \frac{1 - \cos a - \cos b + \cos c}{\sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}}.$$

Similarly, tan $\frac{1}{2}(a+b-c)$ is expressed by the angles.

7. If CD be drawn perpendicular to AB, then will

$$\sin CD = \sin B \sin BC$$

$$= \frac{2}{\sin c}\sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}$$

$$= \frac{2}{\sin C}\sqrt{-\cos S \cos (S-A) \cos (S-B) \cos (S-C)}.$$

8. To find the circular radius of the small circle which may be inscribed in a spherical triangle.

Referring to the diagram of Article (79), *Pl Tr.*, and supposing all the lines employed to be arcs of great circles, we have $Ab = Ac$, $Ba = Bc$, $Ca = Cb$:

$\therefore Ab + Ba + Ca = Ac + Bc + Cb$, which give

$2\,Ab + a = b + c$, $Ab = s - a$, $Bc = s - b$, and $Ca = s - c$:

whence, from $\sin Ab = \tan ob \cot oAb$, we obtain

$$\tan r = \sin (s-a) \tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s-a)(s-b) \sin (s-c)}{\sin s}}.$$

9. To find the circular radius of the small circle which may be circumscribed about a spherical triangle.

If O be the centre of the circumscribed circle, we have $\angle\, BAO = \angle\, ABO$, $\angle\, BCO = \angle\, CBO$, $\angle\, ACO = \angle\, CAO$: whence, $2\angle\, BAO = A + B - C$, and $\angle\, BAO = S - C$: similarly, $\angle\, ACO = S - B$, and $\angle\, CBO = S - A$:

$\therefore\ \cos BAO = \cot AO \tan \frac{1}{2}c$, gives $\tan R$

$$= \sec(S - C) \tan \tfrac{1}{2}c = \frac{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}{\sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}}.$$

From these two Articles, we have immediately

$$\tan r \tan R = \frac{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}{\sin \frac{1}{2}(a + b + c)},$$

which is analogous to the result of Article (80), *Pl. Tr.*

10. Given two sides and the included angle, to find the rectilineal angle formed by the chords of these sides.

Let α, β, γ be the chords of the sides a, b, c:

then, $\cos c = \sin a \sin b \cos C + \cos a \cos b$, gives

$$1 - 2 \sin^2 \tfrac{1}{2}c = 4 \sin \tfrac{1}{2}a \cos \tfrac{1}{2}a \sin \tfrac{1}{2}b \cos \tfrac{1}{2}b \cos C$$
$$+ (1 - 2 \sin^2 \tfrac{1}{2}a)(1 - 2 \sin^2 \tfrac{1}{2}b):$$

$$\therefore\ 1 - \tfrac{1}{2}\gamma^2 = \alpha\beta \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos C + 1 - \tfrac{1}{2}\alpha^2 - \tfrac{1}{2}\beta^2 + \tfrac{1}{4}\alpha^2\beta^2:$$

whence, $\frac{1}{2}(\alpha^2 + \beta^2 - \gamma^2) = \alpha\beta \cos \frac{1}{2}a \cos \frac{1}{2}b \cos C + \frac{1}{4}\alpha^2\beta^2$;

but if C' be the angle contained between α, β, we have

$$\cos C' = \frac{\alpha^2 + \beta^2 - \gamma^2}{2\alpha\beta} = \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos C + \tfrac{1}{4}\alpha\beta$$
$$= \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos C + \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b.$$

If $C' = C - \theta$, and θ be very small, we have

$$\cos C' = \cos (C - \theta) = \cos C \cos \theta + \sin C \sin \theta$$
$$= \cos C + \theta \sin C, \text{ nearly}:$$

whence, $\cos C + \theta \sin C = \{\cos^2 \frac{1}{4}(a + b) - \sin^2 \frac{1}{4}(a - b)\}$
$$\times \cos C + \sin^2 \tfrac{1}{4}(a + b) - \sin^2 \tfrac{1}{4}(a - b);$$

$$\therefore\ \theta = \sin^2 \tfrac{1}{4}(a + b) \tan \tfrac{1}{2}C - \sin^2 \tfrac{1}{4}(a - b) \cot \tfrac{1}{2}C, \text{ nearly}.$$

Hence C' is greater than C when it is an acute angle, but less when it is either an obtuse angle, or a right angle, *both generally* and approximately.

11. Given the chords of two sides of a spherical triangle and their included angle, to find the angle contained by the sides.

Since, $\cos C' = \cos \frac{1}{2}a \cos \frac{1}{2}b \cos C + \sin \frac{1}{2}a \sin \frac{1}{2}b$,

we have $$\cos C = \frac{\cos C' - \sin \frac{1}{2}a \sin \frac{1}{2}b}{\cos \frac{1}{2}a \cos \frac{1}{2}b} = \frac{4 \cos C' - \alpha\beta}{\sqrt{(4-\alpha^2)(4-\beta^2)}}.$$

12. Given the oblique angle contained between two objects above the horizon, to find the corresponding horizontal angle.

In the diagram of Article (8), let a and n be the objects whose angular distance aOn is *observed* from the point O in the *horizon:* that is, if straight lines were drawn to O from a and n, the $\angle\, aOn$ would be the *oblique* angle, and the $\angle\, AON$ the corresponding horizontal angle, which is the same as the spherical angle APN: let $aOn = C$, $AOa = H$, $NOn = h$, then by Article (30), we have

$$\sin^2 \tfrac{1}{2} APN = \frac{\sin \frac{1}{2}(C+H-h) \sin \frac{1}{2}(C+h-H)}{\cos H \cos h}:$$

and thus the horizontal angle AON is determined.

This is technically called the *Reduction to the Horizon:* and in practice, H and h are very small:

whence, if $\angle\, AON = \angle\, aOn + \theta = C + \theta$, we have

$$\cos(C+\theta) = \frac{\cos C - \sin H \sin h}{\cos H \cos h} = \frac{\cos C - Hh}{(1-\frac{1}{2}H^2)(1-\frac{1}{2}h^2)}, \text{ nearly,}$$

$$= (\cos C - Hh)\{1 + \tfrac{1}{2}(H^2 + h^2)\}, \text{ very nearly:}$$

that is, $\cos C - \theta \sin C = \cos C - Hh + \frac{1}{2}(H^2 + h^2)\, C$:

$$\text{which gives } \theta = \frac{1}{\sin C}\{Hh - \tfrac{1}{2}(H^2 + h^2) \cos C\}$$

$$= \frac{(H^2 + 2Hh + h^2) \sin^2 \frac{1}{2}C - (H^2 - 2Hh + h^2) \cos^2 \frac{1}{2}C}{4 \sin \frac{1}{2}C \cos \frac{1}{2}C}$$

$$= \{\tfrac{1}{2}(H+h)\}^2 \tan \tfrac{1}{2}C - \{\tfrac{1}{2}(H-h)\}^2 \cot \tfrac{1}{2}C,$$

an *approximate* value of the difference between the circular measures of the oblique and *horizontal* angles, which must be added to the *former* to obtain the latter.

13. The sides of a spherical triangle being small with respect to the radius of the sphere, it is required to find the approximate values of the angles of a plane triangle whose sides are of the same magnitudes.

If a, b, c be the magnitudes of the sides of a spherical triangle described on the surface of a sphere whose radius is r, their circular measures will be $\frac{a}{r}$, $\frac{b}{r}$, $\frac{c}{r}$;

$$\text{then, } \cos A = \frac{\cos\frac{a}{r} - \cos\frac{b}{r}\cos\frac{c}{r}}{\sin\frac{b}{r}\sin\frac{c}{r}},$$

which, by substituting for $\cos\frac{a}{r}$, $\cos\frac{b}{r}$, &c., their values given in Article (128), *Pl. Tr.* and properly reducing, becomes

$$\cos A = \frac{b^2+c^2-a^2}{2bc} + \frac{a^4+b^4+c^4-2\,(a^2b^2+a^2c^2+b^2c^2)}{24bcr^2},$$

by neglecting higher powers of $\frac{a}{r}$, &c., than the *fourth* : hat is, $\cos A = \cos A' - \frac{bc}{6r^2}\sin^2 A'$, nearly, where A' denotes the *plane* angle corresponding to A: but if $A = A' + \theta$,

$$\text{we have } \cos A' - \theta \sin A' = \cos A' - \frac{bc}{6r^2}\sin^2 A',$$

$$\text{which gives } \theta = \frac{bc}{6r^2}\sin A' = \frac{1}{3r^2}\,\tfrac{1}{2}bc\sin A':$$

whence, $A' = A - \theta = A - \frac{1}{3r^2}$ area of the *plane* triangle:

$$\text{also, from } 180^0 = A' + B' + C' = A + B + C - \frac{1}{r^2}S,$$

$$\text{we have } \frac{1}{r^2}S = \text{the spherical excess} = E:$$

$$\therefore\; A' = A - \tfrac{1}{3}E,\; B' = B - \tfrac{1}{3}E, \text{ and } C' = C - \tfrac{1}{3}E.$$

Hence, a *spherical* triangle under the above-mentioned circumstances may be treated as a *plane* triangle having he *same* sides, and each of its angles *less* by *one third* of he *spherical excess* than the corresponding angle of the proposed triangle, whenever those sides are sufficiently small, compared to the radius of the sphere, to make the retention of higher powers a matter of no consequence in *practical* computations.

These two theorems which have always been much esteemed for their *Elegance*, as well as for the *Accuracy* of their results in *Trigonometrical Surveys* and *Geodetical Operations*, were first given by *M. Legendre*: and the student may see to what extent they have been used, by referring to any treatise on the figure of the Earth, and more particularly to that contained in the Encyclopedia Metropolitana, by the present *Astronomer Royal*.

14. If we suppose the radius r of the sphere to be *indefinitely* increased so that any *finite* portion of its surface may be considered as a *plane*, we shall have $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, the fundamental formula in plane triangles.

Since the arc, its sine and tangent are all *ultimately* equal, by Article (86), *Pl. Tr.*, it may be observed generally that formulæ for *spherical* triangles involving the *sines* and *tangents* only of the sides will be true for *plane* triangles, when for the sines and tangents of the sides are substituted the sides themselves.

15. In a spherical triangle ABC, if CD be drawn from the angle C to meet the opposite side AB in D,

$$\text{then, } \cos CAD = \frac{\cos CD - \cos AC \cos AD}{\sin AC \sin AD}$$

$$= \frac{\cos BC - \cos AB \cos AC}{\sin AB \sin AC} : \text{ whence we have}$$

$$\cos CD = \frac{\sin AD \cos BC + \cos AC (\sin AB \cos AD - \cos AB \sin AD)}{\sin AB}$$

$$= \frac{\sin AD \cos BC + \cos AC \sin (AB - AD)}{\sin AB}$$

$$= \frac{\sin AD \cos BC + \sin BD \cos AC}{\sin AB}.$$

If AB be bisected in D, or $AD = BD = \frac{1}{2}c$, then

$$\cos CD = \frac{\sin \frac{1}{2}c \cos a + \sin \frac{1}{2}c \cos b}{2 \sin \frac{1}{2}c \cos \frac{1}{2}c} = \frac{\cos a + \cos b}{2 \cos \frac{1}{2}c}$$

$$=\frac{\cos\frac{1}{2}(a+b)\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}=\frac{\sin(A+B)}{\sin C}\cos\tfrac{1}{2}c.$$

If CD bisect the angle C, or $\angle ACD=\angle BCD=\frac{1}{2}C$,

ien, $\sin AD=\dfrac{\sin\frac{1}{2}C}{\sin A}\sin CD,\quad \sin BD=\dfrac{\sin\frac{1}{2}C}{\sin B}\sin CD$:

$$\therefore\ \cos CD=\frac{\sin\frac{1}{2}C\sin CD}{\sin AB}\left\{\frac{\cos BC}{\sin A}+\frac{\cos AC}{\sin B}\right\}$$

$$=\frac{\sin\frac{1}{2}C\sin CD}{\sin C}\left\{\frac{\cos a}{\sin a}+\frac{\cos b}{\sin b}\right\}=\frac{\sin CD}{2\cos\frac{1}{2}C}\,\frac{\sin(a+b)}{\sin a\sin b}:$$

$$.\ \tan CD=\frac{2\sin a\sin b}{\sin(a+b)}\cos\tfrac{1}{2}C=\frac{\sin A\sin B\sin c}{2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)\cos\frac{1}{2}C}.$$

Both these results may be expressed in other terms y substitution, and the Student, assisted by Article (14), ·ill be led immediately to the conclusions of Articles 5, 36, *Ap.* I.)

16. If through any point O in the surface of a ›herical triangle, great circles be drawn from the angles ', B, C, to meet the opposite sides in a, b, c; it may be roved as in Article (32. *Ap.* I.), that

$$\sin Ab\sin Bc\sin Ca=\sin Ac\sin Ba\sin Cb:$$

$$\text{n } ABO\sin BCO\sin CAO=\sin CBO\sin ACO\sin BAO:$$

ıd the converse will hold good as in the same Article, hich may be applied to the same instances.

17. The great circles drawn from the angles of any ›herical triangle bisecting the angles, meet in the same ›int O: and if h, k, l, be its circular distances from A, , C, respectively, we have

om triangle AOB, $\cos\frac{1}{2}ABC\sin k=\dfrac{\cos h-\cos k\cos c}{\sin c}$:

om triangle BOC, $\cos\frac{1}{2}ABC\sin k=\dfrac{\cos l-\cos k\cos a}{\sin a}$:

$$\therefore\ \cos h\sin a+\cos k\sin(c-a)-\cos l\sin c=0:$$

imilarly, $\cos k\sin b+\cos l\sin(a-b)-\cos h\sin a=0$;

and $\cos l\sin c+\cos h\sin(b-c)-\cos k\sin b=0$:

$$\sin(b-c)\cos h+\sin(c-a)\cos k+\sin(a-b)\cos l=0.$$

18. If the sides of a spherical triangle ABC be cut by a *transversal* great circle in a, b, c; then it is proved as in Article (16. *Ap.* II.) that

$$\sin Ac \sin Ba \sin Cb = \sin aC \sin bA \sin cB:$$

and when this equality is satisfied, it follows *conversely* that a, b, c lie in a great circle.

19. If from a point A in the surface of a sphere great circles ADE, AFG be drawn to cut any circle of the sphere whose pole is P in D, E and F, G: then if PD, PE, PF, PG be drawn, and PH, PK be perpendicular to ADE, AFG: we have

$$\cos AH \cos HP = \cos AP = \cos AK \cos KP:$$

$$\cos EH \cos HP = \cos PE = \cos GK \cos KP:$$

$$\text{whence, } \frac{\cos AH - \cos EH}{\cos AH + \cos EH} = \frac{\cos AK - \cos GK}{\cos AK + \cos GK}:$$

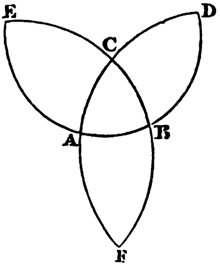
$$\text{or, } \tan \tfrac{1}{2}AD \tan \tfrac{1}{2}AE = \tan \tfrac{1}{2}AF \tan \tfrac{1}{2}AG;$$

which, by Article (14. *Ap.* II.), leads immediately to *Euclid* III. Prop. 35, and 36.

When the points F and G coincide, or AFG *touches* the circle, we have $\tan \frac{1}{2}AD \tan \frac{1}{2}AE = \tan^2 \frac{1}{2}AF$, analogous to *Euclid* III., Prop. 36: and the further coincidence of E with D, gives $AD = AF$, as in *Euclid* III., Prop. 37.

Enquiries of this kind have led to very *elegant* and *striking* results; and though the utility of many of them in any *practical* point of view may be questioned, their discovery nevertheless evinces great dexterity in the management of the symbols employed in this part of *Mathematical Science:* and the student is now referred to *Leybourne's* Mathematical Repository, the *Ladies'* and *Gentleman's* Diaries, the Appendix to *Young's* Trigonometry, &c., in which he will be gratified by many most elegant properties with their demonstrations from the hands of Messrs. *Lowry*, *Whitley*, *Davies*, *Rutherford*, &c.: and the *Annales des Mathematiques*, as well as other *Foreign* periodical journals of science, will shew the interest which such subjects have *generally* excited among scientific men.

20. If the sides of the spherical triangle ABC, be roduced to meet again in D, E, F respectively opposite to A, B, C, it is manifest that three *additional*



riangles will thus be formed: and the *four* triangles aken together have been very appropriately termed the *Associated System of Triangles*, of which ABC is the *Base* or *Fundamental* Triangle.

Hence of the triangle BCD, the sides are manifestly a, $\pi - b$, $\pi - c$, and the angles are A, $\pi - B$, $\pi - C$: and similarly of the others: thus, the triangles, BCD, ACE and ABF may be called the *supplemental triangles* of the associated system of triangles whose base is ABC. See Article (19), of the Spherical Trigonometry.

21. All the formulæ investigated for the triangle ABC, will of course hold good for each of these, when he proper substitutions are made.

Thus, to find the radius of the circle inscribed in he triangle BCD, we have

$$2s_1 = a + \pi - b + \pi - c = 2\pi - (b + c - a), \text{ and } s_1 = \pi - \tfrac{1}{2}(b + c - a):$$

$$\therefore\ 2(s_1 - a_1) = a + \pi - b + \pi - c - 2a = 2\pi - (a + b + c),$$

$$\text{or, } s_1 - a_1 = \pi - \tfrac{1}{2}(a + b + c):$$

$$2(s_1 - b_1) = a + \pi - b + \pi - c - 2\pi + 2b = a + b - c,$$

$$\text{or, } s_1 - b_1 = \tfrac{1}{2}(a + b - c):$$

$$2(s_1 - c_1) = a + \pi - b + \pi - c - 2\pi + 2c = a + c - b,$$

$$\text{or, } s_1 - c_1 = \tfrac{1}{2}(a + c - b):$$

whence, if r_1 be the radius of its inscribed circle, we find

$$\tan r_1 = \frac{\sqrt{\sin s_1 \sin (s_1 - a_1) \sin (s_1 - b_1) \sin (s_1 - c_1)}}{\sin s_1}$$

$$= \frac{\sqrt{\sin (s - a) \sin s \sin (s - c) \sin (s - b)}}{\sin (s - a)}$$

$$= \frac{\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}}{\sin (s - a)}.$$

If the numerator of this expression, which is symmetrical with respect to all the sides, be denoted by n, we have $\tan r_1 = \dfrac{n}{\sin (s - a)}$: and if r_2, r_3 be the corresponding radii for the triangles ACE, ABF, we shall conclude that

$$\tan r_2 = \frac{n}{\sin (s - b)}, \text{ and } \tan r_3 = \frac{n}{\sin (s - c)}:$$

all of which might have been obtained in the same way as r has been found in Article (8. *Ap.* II.): and we can scarcely fail to remark the similarity of these results to what were obtained in Article (43. *Ap.* I.) for a plane triangle.

For the circumscribed triangle, of BCD we have

$$\sin \tfrac{1}{2} a_1 = \sin \tfrac{1}{2} a, \quad \sin \tfrac{1}{2} b_1 = \cos \tfrac{1}{2} b, \quad \text{and } \sin \tfrac{1}{2} c_1 = \cos \tfrac{1}{2} c:$$

$$\therefore \tan R_1 = \frac{2 \sin \frac{1}{2} a_1 \sin \frac{1}{2} b_1 \sin \frac{1}{2} c_1}{\sqrt{\sin s_1 \sin (s_1 - a_1) \sin (s_1 - b_1) \sin (s_1 - c_1)}}$$

$$= \frac{2 \sin \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}{\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}}.$$

$$\text{Similarly, } \tan R_2 = \frac{2 \cos \frac{1}{2} a \sin \frac{1}{2} b \cos \frac{1}{2} c}{\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}},$$

$$\text{and } \tan R_3 = \frac{2 \cos \frac{1}{2} a \cos \frac{1}{2} b \sin \frac{1}{2} c}{\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}}.$$

22. From the last Article are readily obtained

$$\frac{\tan r_1}{\tan r} = \frac{n}{\sin (s - a)} \times \frac{\sin s}{n} = \frac{\sin s}{\sin (s - a)}:$$

$$\frac{\tan r_1}{\tan r_2} = \frac{n}{\sin(s-a)} \times \frac{\sin(s-b)}{n} = \frac{\sin(s-b)}{\sin(s-a)} : \ \&c.:$$

$$\tan r \tan r_1 \tan r_2 \tan r_3 = n^2 :$$

$$\frac{\tan R_1}{\tan R} = \frac{2 \sin \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}{n} \times \frac{n}{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c} = \frac{\cot \frac{1}{2} b}{\tan \frac{1}{2} c} :$$

$$\frac{\tan R_1}{\tan R_2} = \tan \tfrac{1}{2} a \cot \tfrac{1}{2} b : \ \&c.$$

23. By substitutions of little difficulty, we shall have

$$\cot R = \cos(S - C) \cot \tfrac{1}{2} c$$

$$= \frac{\sqrt{-\cos S \cos(S-A) \cos(S-B) \cos(S-C)}}{-\cos S} = \frac{N}{\cos S} :$$

$$\cot R_1 = \frac{N}{\cos(S-A)}, \ \cot R_2 = \frac{N}{\cos(S-B)}, \ \cot R_3 = \frac{N}{\cos(S-C)} :$$

$$\text{and } \therefore \ \cot R \cot R_1 \cot R_2 \cot R_3 = N^2.$$

24. From these are easily derived the following:

$$\sin^2 s = \cot r \tan r_1 \tan r_2 \tan r_3 :$$

$$\sin^2 (s-a) = \tan r \cot r_1 \tan r_2 \tan r_3 : \ \&c.:$$

$$\cos^2 S = \tan R \cot R_1 \cot R_2 \cot R_3 :$$

$$\cos^2 (S-A) = \cot R \tan R_1 \cot R_2 \cot R_3 : \ \&c.:$$

and a great variety of other elegant and remarkable Theorems may be deduced by algebraical combinations of the fundamental expressions above found, which cannot be admitted in this place.

25. From $\tan r = \sin(s-c) \tan \frac{1}{2} C$, we infer that if the *vertical* angle of a triangle and the difference between the *base* and the sum of the *other* two sides be given, the radius of the circle inscribed in the triangle will be *invariable* and the centre a *fixed* point: and a *similar* conclusion will be derived from $\cot R = \cos(S - C) \cot \frac{1}{2} c$, with regard to the circumscribed circle.

26. Since the spherical excess $E = A + B + C - 180^\circ$,

$$\text{we have } \sin \tfrac{1}{2} E = -\cos \tfrac{1}{2} (A + B + C)$$

$$= \sin \tfrac{1}{2} (A + B) \sin \tfrac{1}{2} C - \cos \tfrac{1}{2} (A + B) \cos \tfrac{1}{2} C$$

$$= \frac{1}{2\cos\frac{1}{2}c}\{\cos\tfrac{1}{2}(a-b) - \cos\tfrac{1}{2}(a+b)\}\sin C$$

$$= \frac{\sin\frac{1}{2}a\sin\frac{1}{2}b\sin C}{\cos\frac{1}{2}c}:$$

and this, by putting for $\sin C$ its value in terms of the sides, gives

$$\sin\tfrac{1}{2}E = \frac{\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c},$$

which is adapted to logarithms, and known by the name of *Cagnoli's* Theorem.

27. From $\cot\frac{1}{2}E = \dfrac{\cot\frac{1}{2}a\cot\frac{1}{2}b + \cos C}{\sin C}$,

and $\sin\frac{1}{2}E = \dfrac{\sin\frac{1}{2}a\sin\frac{1}{2}b\sin C}{\cos\frac{1}{2}c}$,

is obtained $\cos\frac{1}{2}E = \dfrac{\cos^2\frac{1}{2}a + \cos^2\frac{1}{2}b + \cos^2\frac{1}{2}c - 1}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}$,

which is *rational*, but not adapted to logarithms.

Hence, we shall also readily prove that

$$\cos\tfrac{1}{2}E = \{\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b + \sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\cos C\}\sec\tfrac{1}{2}c.$$

28. $\text{Tan}\frac{1}{4}E = \dfrac{2\sin\frac{1}{4}(A+B+C-180^\circ)\cos\frac{1}{4}(A+B-C+180^\circ)}{2\cos\frac{1}{4}(A+B+C-180^\circ)\cos\frac{1}{4}(A+B-C+180^\circ)}$

$$= \frac{\sin\frac{1}{2}(A+B) - \cos\frac{1}{2}C}{\cos\frac{1}{2}(A+B) + \sin\frac{1}{2}C} = \frac{\cos\frac{1}{2}(a-b) - \cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b) + \cos\frac{1}{2}c}\cot\tfrac{1}{2}C$$

$$= \frac{\sin\frac{1}{2}(s-a)\sin\frac{1}{2}(s-b)}{\cos\frac{1}{2}s\cos\frac{1}{2}(s-c)}\cot\tfrac{1}{2}C$$

$$= \sqrt{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)}:$$

which was discovered by *Lhuilier*, and is adapted to logarithmic computation

29. Many other expressions of great elegance have been invented for determining the spherical excess in terms of both the sides and angles of the triangle: and *each of* them may by the proper substitutions be used for *the* associated triangles described in Article (20. Ap. II.)

Thus, if E_1 denote the excess for the triangle BCD, $\tan\frac{1}{4}E_1=\sqrt{\cot\frac{1}{2}s\cot\frac{1}{2}(s-a)\tan\frac{1}{2}(s-b)\tan\frac{1}{2}(s-c)}$: &c.: and it will readily follow that

$$\tan\tfrac{1}{4}E\tan\tfrac{1}{4}E_1\tan\tfrac{1}{4}E_2\tan\tfrac{1}{4}E_3$$

$$=\cot\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c):$$

$$\tan^2\tfrac{1}{2}s=\tan\tfrac{1}{4}E\cot\tfrac{1}{4}E_1\cot\tfrac{1}{4}E_2\cot\tfrac{1}{4}E_3:$$

$$\cot^2\tfrac{1}{2}(s-a)=\cot\tfrac{1}{4}E\tan\tfrac{1}{4}E_1\cot\tfrac{1}{4}E_2\cot\tfrac{1}{4}E_3:\ \&c.:$$

$$\cos S=-\sin\tfrac{1}{2}E:\ \cos(S-A)=\sin\tfrac{1}{2}E_1:\ \&c.:$$

$$\sin\tfrac{1}{2}E\sin\tfrac{1}{2}E_1\cot^2\tfrac{1}{2}a=\sin\tfrac{1}{2}E_2\sin\tfrac{1}{2}E_3:\ \&c.$$

30. If S be the surface of a spherical triangle each of whose angles is 120^0, and S' that of its polar triangle:

$$\text{then } S=180^0 \text{ and } \therefore \tan\tfrac{1}{3}S=\sqrt{3}:$$

$$\text{also, } S'=A'+B'+C'-180^0=3\cos^{-1}\tfrac{1}{3}-180^0,$$

$$\text{and } \tfrac{1}{3}S'=\cos^{-1}\tfrac{1}{3}-60^0=\tan^{-1}2\sqrt{2}-60^0,$$

$$\text{or, } \tan\tfrac{1}{3}S'=\frac{2\sqrt{2}-\sqrt{3}}{1+2\sqrt{6}}:$$

$$\text{whence, } \frac{\tan\frac{1}{3}S}{\tan\frac{1}{3}S'}=\frac{\sqrt{3}(1+2\sqrt{6})}{2\sqrt{2}-\sqrt{3}}=\frac{6\sqrt{2}+\sqrt{3}}{2\sqrt{2}-\sqrt{3}}.$$

31. If C be the right angle of the triangle ABC, we know that $\cos A=\cos a\sin B$: whence, if a remain *constant*, and δA, δB be the *corresponding* small *variations* in A and B, we have $\cos(A+\delta A)=\cos a\sin(B+\delta B)$, or $\cos A\cos\delta A-\sin A\sin\delta A=\cos a(\sin B\cos\delta B+\cos B\sin\delta B)$, wherein δA, δB being considered *very small*, or $\cos\delta A=1$, $\sin\delta A=\delta A$, &c., it follows that

$$\cos A-\sin A\,\delta A=\cos a(\sin B+\cos B\,\delta B),$$

$$\text{and } \therefore \frac{\delta A}{\delta B}=-\frac{\cos a\cos B}{\sin A}=-\frac{\cos A\cos B}{\sin A\sin B}=-\frac{\cot B}{\tan A}:$$

that is, the *increment* of A will be to the *decrement* of B *very nearly*, in the ratio of $\cot B$ to $\tan A$, when the changes in the variable elements of the triangle are *small*.

Again, from $\cos c=\cos a\cos b$,

we have $\cos(c+\delta c)=\cos a\cos(b+\delta b)$,

which from the considerations above, gives

$\cos c-\sin c\,\delta c=\cos a(\cos b-\sin b\,\delta b)=\cos a\cos b-\cos a\sin b\,\delta b$:

$$\therefore \sin c\delta c = \cos a \sin b\delta b, \text{ and}$$

$$\frac{\delta c}{\delta b} = \frac{\cos a \sin b}{\sin c} = \frac{\cos c \sin b}{\cos b \sin c} = \frac{\tan b}{\tan c}.$$

The same kinds of processes depending upon Articles (190)—(196) *Pl. Tr.*, will lead to the ratios of the *small changes* in *any* two parts of the triangle, when *two* of the other parts remain *invariable.*

32. In any triangle, whereof A and a remain constant,

$$\text{from } \cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}, \text{ we shall have}$$

$$\frac{\delta B}{\delta C} = -\frac{\sin B \cos C \cos a + \cos B \sin C}{\cos B \sin C \cos a + \sin B \cos C} = -\frac{\cos b}{\cos c},$$

by eliminating B and C.

If A and B remain constant, $\cos a = \dfrac{\cos A + \cos B \cos C}{\sin B \sin C}$

$$\text{gives } \frac{\delta C}{\delta a} = \frac{\sin a \sin B \sin^2 C}{\cos B + \cos A \cos C} = \frac{\sin C}{\cot b}.$$

$$\text{From } \cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B}, \text{ we shall have}$$

$$\frac{\delta C}{\delta c} = \frac{\sin A \sin B \sin c}{\sin C} = \sin a \sin B:$$

$$\text{also, } \frac{\delta a}{\delta b} = \frac{\tan a}{\tan b}, \text{ and } \frac{\delta a}{\delta c} = \frac{\cos b \sin A}{\sin C}.$$

Let A and b remain constant; then it will be found by the same process and on the same hypothesis, that

$$\frac{\delta B}{\delta C} = -\frac{\cos A + \cos B \cos C}{\sin B \sin C} = -\cos a:$$

$$\frac{\delta B}{\delta a} = -\frac{\sin B \cos a}{\cos B \sin a} = -\frac{\tan B}{\tan a}:$$

$$\frac{\delta a}{\delta c} = \frac{\cos b - \cos a \cos c}{\sin a \sin c} = \cos B:$$

$$\frac{\delta C}{\delta c} = \frac{\tan B \cos B}{\tan a \cos a} = \frac{\sin B}{\sin a}.$$

If *a* and *b* remain unchanged, we shall have

$$\frac{\delta A}{\delta B} = \frac{\tan A}{\tan B} : \quad \frac{\delta C}{\delta c} = \frac{1}{\sin a \sin B} :$$

$$\frac{\delta A}{\delta c} = -\frac{\cot B}{\sin c}, \text{ and } \frac{\delta A}{\delta C} = -\frac{\cos B \sin a}{\sin c}.$$

These results are all derived from the formulæ eferred to in the last Article, which necessarily imply hat they are mere *approximations:* in their *ultimate* ondition, they are *absolutely* correct, to understand hich, a knowledge of the *Differential Calculus* or the *octrine of Limits* will be required; but their *practical* nportance in *Astronomy* and in other parts of science in s present advanced state, will scarcely fail to be recognized.

This subject was first treated of by *Roger Cotes, lumian Professor* at *Cambridge,* in his tract entitled *stimatio Errorum in mixtâ Mathesi,* &c., which is the rst of his *Opera Miscellanea,* and may be found at the nd of the *Harmonia Mensurarum.*

CHAPTER II.

MISCELLANEOUS THEOREMS FOR PRACTICE.

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1. In a right-angled spherical triangle, prove that

$$\frac{\sin(a-b)}{\sin(a+b)} = \frac{\tan\frac{1}{2}(A+B)}{\cot\frac{1}{2}(A-B)}: \ \tan^2\tfrac{1}{2}a = \frac{\tan\frac{1}{2}(c+b)}{\cot\frac{1}{2}(c-b)}:$$

$$\tan^2(45^\circ+\tfrac{1}{2}c) = \frac{\tan\frac{1}{2}(B+b)}{\tan\frac{1}{2}(B-b)}: \ \tan^2(45^\circ+\tfrac{1}{2}A) = \frac{\cot\frac{1}{2}(B+b)}{\tan\frac{1}{2}(B-b)}.$$

$$\frac{\text{Sin}(c-b)}{\sin(c+b)} = \tan^2\tfrac{1}{2}A: \ \tan^2\tfrac{1}{2}a = \frac{\tan\frac{1}{2}(B+A-90^\circ)}{\tan\frac{1}{2}(B-A+90^\circ)}:$$

$$2\cos c = \cos(a+b) + \cos(a-b).$$

3. In any spherical triangle, it is required to prove that $\cos^2\frac{1}{2}c\sin(A+B) = \cos\frac{1}{2}(a+b)\cos\frac{1}{2}(a-b)\sin C$: and $\sin^2\frac{1}{2}c\sin(A-B) = \sin\frac{1}{2}(a+b)\sin\frac{1}{2}(a-b)\sin C$.

4. $$\text{Sin}^2 s = \frac{(\sin^2 a\sin^2 b\sin^2 c\sin A\sin B\sin C)^{\frac{1}{3}}}{2\tan\frac{1}{2}A\tan\frac{1}{2}B\tan\frac{1}{2}C}:$$

$$\cos^2 S = \frac{(\sin a\sin b\sin c\sin^2 A\sin^2 B\sin^2 C)^{\frac{1}{3}}}{2\cot\frac{1}{2}a\cot\frac{1}{2}b\cot\frac{1}{2}c}.$$

5. If O be a point in a triangle, and Oa, Ob, Oc be at right angles to the sides BC, AC, AB: then will

$$\cos Ac\cos Ba\cos Cb = \cos Ab\cos Bc\cos Ca.$$

6. If two great circles PaA, PbB intersect two other great circles QBA, Qba, in the points A, B and a, b:

$$\text{then will } \frac{\sin AQ}{\sin BQ} = \frac{\sin Aa\sin Pb}{\sin Bb\sin Pa}.$$

7. If two great circles ABC, abc be cut by three hers which pass through the same point, in A, B, C nd a, b, c: then will

$$\frac{\sin Bb}{\sin Pb} \sin AC = \frac{\sin Aa}{\sin Pa} \sin BC + \frac{\sin Cc}{\sin Pc} \sin AB.$$

8. If O be the pole of the circumscribed circle of he triangle ABC, then will

$$\cos \tfrac{1}{2} AOB = \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos C + \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b.$$

9. In a spherical triangle ABC, if AB be bisected n D, and $C = A + B$, then will $CD = AD = BD$.

10. If in a right-angled triangle α, β be the arcs rawn from the right angle respectively perpendicular c, and bisecting the hypothenuse c; prove that

$$\sin \tfrac{1}{2} c = \sin \beta \div \sqrt{1 + \sin^2 \alpha}.$$

11. The arc of a great circle which joins the middle f the side of a triangle with its pole, bisects the angle ormed by the arcs, which join the pole with the poles of he other sides.

12. If arcs drawn from the angles of a triangle hrough any point O, meet the opposite sides in a, b, c: nd P be the pole of the circumscribed circle, then

$$\frac{\sin Oa}{\sin Aa} + \frac{\sin Ob}{\sin Bb} + \frac{\sin Oc}{\sin Cc} = \frac{\cos OP}{\cos R}.$$

13. If ABC be an equilateral triangle, P the pole f its circumscribed circle: then if Q be any point on he sphere,

$$\cos QA + \cos QB + \cos QC = 3 \cos PA \cos PQ.$$

14. Retaining the notation before used, prove that

$$\cot r_1 + \cot r_2 + \cot r_3 - \cot r = 2 \tan R.$$

15. If D be the distance between the centres of the nscribed and circumscribed circles whose radii are R, r: hen $\sin^2 D = \sin^2 (R - r) - \cos^2 R \sin^2 r$.

16. If D be the distance between the centres of the circles whose radii are R and r_1: then

$$\sin^2 D = \sin^2 (R + r_1) - \cos^2 R \sin^2 r_1.$$

17. The sums of the opposite angles of a spherical quadrilateral inscribed in a circle, are equal to each other.

18. In the same figure, the product of the sines of the semidiagonals is equal to the sum of the products of the sines of the halves of the opposite sides.

19. If a, b, c, d be the sides and D be the diagonal joining A and C, then will

$$\sin^2 \tfrac{1}{2} D = \frac{(\sin\frac{1}{2}a \sin\frac{1}{2}d + \sin\frac{1}{2}b \sin\frac{1}{2}c)(\sin\frac{1}{2}a \sin\frac{1}{2}c + \sin\frac{1}{2}b \sin\frac{1}{2}d)}{\sin\frac{1}{2}a \sin\frac{1}{2}b + \sin\frac{1}{2}c \sin\frac{1}{2}d}.$$

20. If a_1, a_2, a_3, a_4; d_1, d_2 be the sides and diagonals of any spherical quadrilateral: and if k be the arc joining the middle points of the diagonals, then will

$$\cos k = \frac{\cos a_1 + \cos a_2 + \cos a_3 + \cos a_4}{4 \cos \frac{1}{2} d_1 \cos \frac{1}{2} d_2}.$$

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